

National Council of Teachers of Mathematics SPA Standards—Secondary

1. Brief Introduction to the program standards

Attached are the 2019 revised NCTM standards for the preparation of secondary teachers of mathematics. The NCTM standards have been modified and streamlined, with the content preparation of teachers reflecting the major areas of study of secondary mathematics. Increased attention has been given to equity, rigorous mathematics understanding for secondary students, and the development of positive mathematical identities.

This revision of the NCTM standards reflects recent and current conversations in the mathematics community regarding not just standards of content, but the deepening urgency to address the nature of effective mathematics teaching and learning. These standards attend to current leadership documents for the preparation of mathematics teachers and the increasing calls for reform of high school mathematics including policies, curriculum, and teaching practices. The standards also take into account the need to prepare teachers to engage students to achieve and succeed with respect to relevant college and career ready standards.

The revised NCTM standards are informed by multiple factors including the complementary work of other organizations who are concerned about the preparation of mathematics teachers, identification of critical changes needed in secondary education, and changed within the CAEP system. Goals of the NCTM Task Force charged with leading the revision included consideration of the challenges some experienced with previous standards from the perspectives of institutions developing reports, and report reviewers and auditors. In addition, the Task Force considered other recently approved CAEP standards to work towards consistency across programs of study. While some standards are designed to be strictly aspirational, the revised NCTM standards are designed to be attainable while strengthening mathematics education programs in providing evidence of high-quality teacher education for secondary mathematics teachers.

Structurally, the adoption of the 2012 NCTM standards moved from 17 standards to 7. A maximum of 8 assessments were allowed and included a required content licensure exam and the student teaching assessment. In developing the 2012 NCTM standards, national dialogue was strongly focused on the importance of deep content knowledge for teachers. The mathematics competencies for standard 1 in number, algebra, geometry, probability and statistics, calculus, and discrete mathematics as well as the mathematical processes described in standard 2 reflected the mathematics community's specific and detailed responses. Revised NCTM standards 3-6 provided expanded standards and elements that focused on effective mathematics teaching that engages students in reasoning, communicating, analyzing and applying the mathematics they were learning. Because the 2012 NCTM standards and elements existed as a stand-alone document, the elements became quite dense in an effort for clarity and emphasis.

In retrospect, that the level of specificity in the 2012 NCTM standards has made the designing of assessments and subsequent alignment with the NCTM standards and elements a more cumbersome and disconnected process. In many cases, the level of specificity required extensive development of assessments that typically extended beyond what a beginning teacher is able to demonstrate and achieve.

The current version of NCTM standards and components (2019) represent a more holistic focus on mathematics content and processes, and an expanded description of the teaching practices as articulated by NCTM and the mathematics community over the last 7 years. The 2019 NCTM standards, components, and rubrics strive to ensure that the components and rubrics reflect the language of the standards and make explicit what candidate competencies would be required for a program to receive national recognition. We will continue to complete the application throughout the spring as we seek feedback from our professional colleagues in NCTM, AMTE, and mathematics education program faculty.

Proposed Standards:

Preamble:

The 2019 revision of the NCTM standards for the preparation of teachers have been developed to reflect the current climate and research regarding the teaching of mathematics. This revision reflects current conversation in the mathematics and mathematics education communities regarding not just standards of content, but the deepening urgency to address the nature of effective mathematics teaching and the goals for mathematical learning for each and every student. These standards attend to current proposals for the preparation of teachers and the increasing calls for reform of the high school mathematics curriculum and teaching practices. They also take into account the need to prepare teachers to engage students with college and career ready standards.

These standards build upon the work of many others who have sought to ensure that the power and beauty of mathematics are made fully evident throughout each student's high school and higher education experiences. Guidance for this particular revision relied on MET II (2012, CBMS), Guidelines for Assessment and Instruction in Statistics Education (GAISE) (American Statistical Association, 2005), Statistical Education of Teachers (ASA) (American Statistical Association, 2005) Common Core State Standards for Mathematics (National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010), Guidelines for Assessment and Instruction in Mathematical Modeling Education (GAIMME) (SIAM, 2016), K-6 Elementary Teacher Standards (CAEP, 2018), and the Standards for the Preparation of Literacy Professionals 2017 (International Literacy Association, 2018). Authors of this revision included members of the team that developed the AMTE Standards for Preparing Teachers of Mathematics (2017) and brought perspective from the development of those aspirational standards.

Using the best of what we know of the importance of the teacher in the individual classroom and the repertoire of knowledge, skills, and commitments needed to teach, the revised NCTM standards are heavily influenced by the growing consensus highlighted in *Principles to Actions: Ensuring Mathematical Success for All* (NCTM, 2014). These principles: Teaching and Learning, Access and Equity, Curriculum, Tools and Technology, Assessment, and Professionalism, help articulate the standards and components developed for those preparing beginning teachers and supervisors of mathematics.

While there have been successes in increasing mathematics achievement as measured by NAEP, SAT, ACT and AP exams, many challenges remain for many students, including many from underrepresented groups and students with disabilities. It is imperative that new teachers provide each and every student with access to high quality mathematics instruction that focuses on meaning, understanding, reasoning and the application of procedural knowledge (NCTM, 2014).

Ongoing challenges for the teaching profession and the continuing challenges of attracting able candidates to teach mathematics have created an urgency to refine the standards for beginning teachers. Accountability for student achievement is reflected in more specific standards and indicators regarding assessment. Candidates will enter schools where “fewer than 50 percent of U.S. high school graduates in 2016 were considered ready for college-level mathematics work” (NCTM, 2018, p. xii) and where these statistics demonstrate the limits in the personal and professional opportunities for students.

Standard 1: Knowing and Understanding Mathematics

Candidates demonstrate and apply, with the incorporation of mathematics technology, conceptual understanding, procedural fluency, and factual knowledge of major mathematical domains: Number; Algebra and Functions; Statistics and Probability; Geometry, Trigonometry, and Measurement; Calculus; and Discrete Mathematics.

Supporting Explanation:

Standard 1 requires that secondary mathematics teacher education candidates (referred to as candidates subsequently) be knowledgeable about the mathematics content they may be responsible to teach. The candidates will be able to establish this knowledge by demonstrating and applying conceptual understanding, procedural fluency, and factual knowledge among the major mathematical domains: Number; Algebra and Functions; Statistics and Probability; Geometry, Trigonometry, and Measurement; Calculus; and Discrete Mathematics.

According to the National Research Council (2001) effective programs of teacher preparation support future teachers to understand the mathematics they teach, how their students learn that mathematics, and how to facilitate student learning. In *Adding it Up: Helping Children Learn Mathematics* (2001), the National Research Council identifies the mathematical proficiencies. “Recognizing that no term captures completely all aspects of expertise, competence, knowledge, and facility in mathematics, we have chosen mathematical proficiency to capture what we believe is necessary for anyone to learn mathematics successfully. Mathematical proficiency, as we see it, has five components, or strands:

- conceptual understanding—comprehension of mathematical concepts, operations, and relations
- procedural fluency—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
- strategic competence—ability to formulate, represent, and solve mathematical problems

- adaptive reasoning—capacity for logical thought, reflection, explanation, and justification
- productive disposition—habitual inclination to see mathematics assessable, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy.” (p. 5)

As one can see from the recommendation, content knowledge is one of the core components of effective teaching of mathematics. To this point, the Association of Mathematics Teacher Educators (2017) emphasizes that a “well-prepared” beginning teacher of mathematics attain a robust understanding of mathematics. The AMTE standards highlight that a mathematics teacher entering the profession must possess both the underlying mathematics and statistical content knowledge for teaching as well as engage in the mathematical and statistical practices that have been highlighted in other documents including essential mathematics concepts outlined in *Catalyzing Change in High School Mathematics* (NCTM, 2018), college and career readiness standards (e.g., Common Core State Standards for Mathematics).

To deepen conceptual understanding, candidates will be looking forward and backward, challenging the knowledge of mathematics that they bring to their college career (CBMS, 2012) as well as acquiring deeper and more complex understanding of the mathematical domains. By revisiting domains of mathematics and expanding their knowledge, these standards require that candidates deepen their conceptual understanding (moving beyond procedural fluency) and build their confidence in the modeling and applications of the mathematics that they are learning. Enhanced problem-solving skills and further experience with multiple representations and models are emphasized as well. This standard is designed to guide best practice in the preparation of mathematics teachers who will have a mission to make accessible and facilitate the learning of high-quality mathematics for all students, first, through their own deep understanding of mathematics.

Selecting Evidence for Standard 1:

Evidence for Standard 1 includes any State-required mathematics content licensure tests aligned to mathematical domain competencies of *NCTM CAEP Mathematics Content for Secondary* accompanied by completer performance data from a minimum of two academic years for an initial report or a minimum of one academic year for a response to conditions or revised report. Performance data must include at minimum mean and range or standard deviation values.

Many programs use a detailed analysis of course grades or transcript analysis (for post-baccalaureate programs) to provide evidence of candidates’ demonstration of content and mathematical processes. Guidelines for the use of course grades are outlined on the NCTM website. Content-based assessment such as projects, course portfolio, or other course products aligned to mathematical domain competencies of *NCTM CAEP Mathematics Content for Secondary* accompanied by candidate performance data can also provide data for Standards 1 and 2.

1a) Essential Concepts in Number. Candidates demonstrate and apply conceptual understanding, procedural fluency, and factual knowledge of number including flexibly applying procedures, and using real and rational numbers in contexts, developing solution strategies and evaluating the correctness of conclusions. Major mathematical concepts in Number include number theory; ratio, rate, and proportion; and structure, relationships, operations, and representations.

<p>Level 1 The Beginning Candidate Candidate does not demonstrate conceptual understanding, procedural fluency, and/or factual knowledge for major mathematical concepts in Number and Operations.</p>	<p>Level 2 The Developing Candidate Candidate demonstrates conceptual understanding, procedural fluency, and/or factual knowledge, but is not able to apply the major mathematical concepts in Number and Operations.</p>	<p>Level 3 The Competent Candidate Candidate demonstrates and applies conceptual understanding, procedural fluency, and factual knowledge of major mathematical concepts in Number.</p> <p>Candidate uses technology to enhance their learning of Number and Operations.</p>	<p>Level 4 The Accomplished Candidate Candidate demonstrates and applies conceptual understanding, procedural fluency, and factual knowledge of major mathematical concepts in Number.</p> <p>Candidate uses technology to enhance their learning of Number and Operation.</p> <p>Candidate makes connections within and among mathematical domains.</p>
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Supporting Explanation:

As prospective mathematics teachers begin their preparation, they typically have had success learning mathematics themselves and have confidence in their ability to learn. In the past, most candidates begin their journey through the calculus sequence and through higher mathematics. Teachers of secondary mathematics often operate with the beliefs that secondary students come with a fully operational understanding of whole numbers, operations, and the types of situations that these topics represent. Yet their own mathematical understanding is often incomplete with the “often unstated assumption of high school mathematics that the real numbers exist and satisfy the same properties of operations as the rational numbers. Teachers need to know how to prove what is unstated in high school in order to avoid false simplifications and to be able to answer questions from students seeking further understanding” (CBMS, 2012, p.60). While procedural fluency is often emphasized in their earlier mathematics experiences, the exploration of university mathematics includes reexamining essential concepts of number in the context of higher levels of mathematics.

The re-exploration of the number system (from early number through the real number system) ensures that candidates will have a fully operational conceptual understanding as well as procedural knowledge of the entire real number system, including rational, irrational numbers with multiple representations. (CBMS, 2012). Extended experience with the use of numbers in multiple contexts including those that represent units, necessitates the ability to use strategies to select current units, scalings, and determine appropriate levels of accuracy (NCTM, 2018). Candidates are able to re-explore number theory with more sophisticated insights and understandings.

Candidates should be able to move fluidly between multiple representations of number and understand how concrete materials and technology can assist in the development of conceptual knowledge. Candidates can use tools from sketching through using modeling technology to explore how these representations can enhance and broaden their understanding. The re-exploration of number provides candidates the opportunity to revisit the learning progressions that come with the building of a number system from countable to infinite, continuous number systems. Candidates are engaged with properties of number systems and explore the differences in the properties of rational numbers and real numbers. The set of real numbers and its subsets are explored looking at the structures of countable and continuous number systems.

As candidates gain facility with each number system, they should also be engaged in persevering with problems that apply the various number systems to real-life, engaging problem solving.

To be prepared to develop student mathematical proficiency, all secondary mathematics teachers should know the following topics related to number based on the Essential Concepts from *Catalyzing Change in High School Mathematics* (NCTM, 2018):

Essential Concepts in Number

- Together, irrational numbers and rational numbers complete the real number system, representing all points on the number line.
- Quantitative reasoning includes, and mathematical modeling requires, attention to units of measurement

1b) Essential Concepts in Algebra and Functions. Candidates demonstrate and apply understandings of major mathematics concepts, procedures, knowledge, and applications of algebra and functions including how mathematics can be used systematically to represent patterns and relationships including proportional reasoning, analyze change, and model everyday events and problems of life and society. Essential Concepts in Algebra and Functions include algebra that connects mathematical structure to symbolic, graphical, and tabular descriptions; connecting algebra to functions; and develops families of functions as a fundamental concept of mathematics. Additional Concepts should include algebra from a more theoretical approach including relationship between structures (e.g., groups, rings, and fields) as well as formal structures for number systems and numerical and symbolic calculations.

Level 1 The Beginning Candidate	Level 2 The Developing Candidate	Level 3 The Competent Candidate	Level 4 The Accomplished Candidate
<p>Candidate does not demonstrate conceptual understanding, procedural fluency, and/or factual knowledge for major mathematical concepts in Algebra and Functions.</p>	<p>Candidate demonstrates conceptual understanding, procedural fluency, and/or factual knowledge, but is not able to apply the major mathematical concepts in Algebra and Functions.</p>	<p>Candidate demonstrates and applies understandings of major mathematics concepts, procedures, knowledge, and applications of algebra and functions.</p> <p>Candidate uses technology to enhance their learning of Algebra and Functions.</p>	<p>Candidate demonstrates and applies conceptual understanding, procedural fluency, and factual knowledge of major mathematical concepts in Algebra and Functions.</p> <p>Candidate uses technology to enhance their learning of Number.</p> <p>Candidate makes connections within and among mathematical domains.</p>

A primary goal in the preparation of a secondary mathematics education teacher candidate is the development of conceptual understanding, procedural fluencies, and mathematical reasoning skills tailored to their future work as teachers as well as examining the transition and connections between the mathematics studied in middle school and high school (CBMS, 2012). The exploration of algebra is one area of mathematics during the transition is especially challenging for the students the candidates will teach (National Research Council, 2001). Although there has been a push for “algebra for all” students, this push has also highlighted the challenges that teacher candidates must be prepared for when engaging in the reasoning and sense making of the mathematics they will teach (CBMS, 2012).

Candidates demonstrate a strong conceptual understanding, procedural fluency, and factual knowledge with respect to the algebra in high school as a systematic exploration and extension of the study of number and operations.

It is useful to distinguish two aspects of algebra that underlie all others: (a) algebra as a systematic way of expressing generality and abstraction, including algebra as generalized arithmetic; and (b) algebra as syntactically guided transformations of symbols.³ These two main aspects of algebra have led to various activities in school algebra, including representational activities, transformational (rule-based) activities, and generalizing and justifying activities.⁴ (CBMS, 2012)

The properties of algebra (e.g., distributive property) can and should be connected to reinforce the fundamental properties rooted in number. Furthermore, a strong understanding and application of the properties allow for the exploration of both unknown or variable quantities as well as various methods to determine the relationships between or among them. Additionally, algebra is linked to functions, which model a dependent relationship between one quantity and another. Together, algebra and functions constitute the language of generalization allowing for the systematic representation of patterns and relationships among numbers and objects, analyze change, and model real world events (NCTM, 2018).

After calculus, linear algebra is the most powerful, comprehensive theory that teachers will encounter. It is an excellent place to begin proving theorems because of the computational nature of many of its proofs and provides an opportunity for teachers to experience the mathematical practice of abstracting a mathematical idea from many examples. A concrete course anchored in specific examples and contemporary applications is more likely to serve the needs of prospective teachers than a course in the theory of abstract vector spaces. Important examples include \mathbb{R}^n and the vector space of polynomials on which differentiation and integration act as linear operators; contemporary applications such as regression, computer visualization, and web search engines (CBMS, 2012, p. 57).

An advanced standpoint reveals much of high school mathematics as the algebra of rings and fields. Abstract algebra for prospective high school teachers should therefore emphasize rings and fields over groups. These structures underlie the base-ten arithmetic of integers and decimals, and operations with polynomials and rational functions. (CBMS, 2012, p.73)

To be prepared to develop student mathematical proficiency, all secondary mathematics teachers should know the following topics related to number based on the Essential Concepts from *Catalyzing Change in High School Mathematics* (NCTM, 2018):

Essential Concepts in Algebra

- Expressions can be rewritten in equivalent forms by using algebraic properties, including properties of addition, multiplication, and exponentiation, to make different characteristics or features visible.
- The structure of an equation or inequality can be purposefully analyzed to better understand context and relationships, and determine efficient strategies to find, justify, and interpret solutions.
- Expressions, equations, and inequalities can be used to analyze and make predictions, both within mathematics and as mathematics is applied in different contexts—in particular, contexts that arise in relation to linear, quadratic, exponential, logarithmic, and trigonometric situations.

Essential Concepts in Connecting Algebra to Functions

- Functions shift the emphasis from a point-by-point relationship between two variables (input/output) to considering an entire set of ordered pairs (where each first element is paired with exactly one second element) as an entity with its own features and characteristics.
- Graphs can be used to obtain exact or approximate solutions of equations, inequalities, and systems of equations and inequalities—including systems of linear equations in two variables and systems of linear and quadratic equations (given or obtained by using technology).

Essential Concepts in Functions

- Functions can be described by using a variety of representations: mapping diagrams, function notation (e.g., $f(x) = x^2$), recursive definitions, tables, and graphs.
- Families of functions have distinguishing attributes (structure) common to all members and that are identifiable in the various representations such as key features of the graphs, including zeros, intercepts, and, when relevant, rate of change, and maximum/minimum values, can also be associated with and interpreted in terms of the equivalent symbolic representations.
- Functions can describe a wide variety of real situations and can help students understand the processes of making and changing assumptions, assigning variables, and finding solutions to contextual problems.

1c) Essential Concepts in Calculus

Candidates demonstrate and apply understandings of major mathematics concepts, procedures, knowledge, and applications of calculus including the mathematical study of the calculation of instantaneous rates of change and the summation of infinitely many small factors to determine some whole. *Essential Concepts in Calculus include limits; continuity; Fundamental Theorem of Calculus; and the meaning and techniques of differentiation and integration.*

Level 1 The Beginning Candidate	Level 2 The Developing Candidate	Level 3 The Competent Candidate	Level 4 The Accomplished Candidate
Candidate does not demonstrate conceptual understanding, procedural fluency, and/or factual knowledge for major mathematical concepts in Calculus.	Candidate demonstrates conceptual understanding, procedural fluency, and/or factual knowledge, but is not able to apply the major mathematical concepts in Calculus.	Candidate demonstrates and applies understandings of major mathematics concepts, procedures, knowledge, and applications of calculus Candidate uses technology to enhance their learning of calculus.	Candidate demonstrates and applies conceptual understanding, procedural fluency, and factual knowledge of major mathematical concepts in calculus. Candidate uses technology to enhance their learning of calculus. Candidate makes connections within and among mathematical domains.

Supporting Explanation:

The exploration of calculus has the potential to unify a great many mathematical concepts studied during high school (CBMS, 2012, p.57). Furthermore, because of its range of applications in a wide variety of fields, the importance of calculus as a topic of study in mathematics must be acknowledged, however, it should be noted that a “rush to calculus” is not advocated (Bressoud, Camp, and Teague, 2012; NCTM, 2018) rather, this study can informally begin when students explore its numerical underpinnings to investigate variation; such as accumulation and rates of change during middle school, high school and beyond (NRC, 2001). For instance, for the majority of students, the content studied in middle school includes the underlying topics (e.g., rates of change, linear equations, sequences, series, areas of plane figures, and volumes of solids) further developed during calculus (CBMS, 2012) whereas *Catalyzing Change in High School Mathematics* (NCTM, 2018) highlights integrated pathways that address concepts related to functions including linear, quadratic, exponential, as well as logarithmic, trigonometric functions and precalculus concepts (e.g., composition of functions, inverse functions) for all students.

Based upon these recommendations the preparation of a secondary mathematics education teacher candidate should include the following calculus in their conceptual understanding and procedural fluency supported by appropriate technology and varied representational tools, including concrete models:

- Limits, continuity, rates of change, the Fundamental Theorem of Calculus, and the meanings and techniques of differentiation and integration
- Parametric, polar, and vector functions
- Sequences and series
- Multivariate functions
- Applications of function, geometry, and trigonometry concepts to solve problems involving calculus

1d) Essential Concepts in Statistics and Probability. Candidates demonstrate and apply understandings of statistical thinking and the major concepts, procedures, knowledge, and applications of statistics and probability including how statistical problem solving and decision making depend on understanding, explaining, and quantifying the variability in a set of data to make decisions. They understand the role of randomization and chance in determining the probability of events. *Essential Concepts in Statistics and Probability include quantitative literacy; visualizing and summarizing data; statistical inference; probability; and applied problems.*

Level 1 The Beginning Candidate	Level 2 The Developing Candidate	Level 3 The Competent Candidate	Level 4 The Accomplished Candidate
<p>Candidate does not demonstrate an understanding of statistical thinking nor the conceptual understanding, procedural fluency, and/or factual knowledge for major mathematical concepts of Statistics and Probability.</p>	<p>Candidate demonstrates an understanding of statistical thinking and conceptual understanding, procedural fluency, and/or factual knowledge, but is not able to apply the major mathematical concepts of Statistics and Probability.</p>	<p>Candidate demonstrates an understanding of statistical thinking, conceptual understanding, procedural fluency, and/or factual knowledge and applies understandings of major mathematics concepts of Statistics and Probability.</p> <p>Candidate uses technology to enhance their learning of Statistics and Probability.</p>	<p>Candidate demonstrates an understanding of statistical thinking, conceptual understanding, procedural fluency, and/or factual knowledge and applies understanding of major mathematical concepts of Statistics and Probability.</p> <p>Candidate uses technology to enhance their learning of Statistics and Probability.</p> <p>Candidate makes connections within and among mathematical domains.</p>

Supporting Explanation:

Candidates should understand how statistics differs from mathematics, particularly the focus on variability and the critical importance of context. The study and application of statistics is grounded in anticipating and accounting for variability in data. “Whereas mathematics answers deterministic questions, statistics provides a coherent set of tools for dealing with “the omnipresence of variability” (Cobb and Moore, 1997)—natural variability in populations, induced variability in experiments, and sampling variability in a statistic, to name a few.” (ASA, 2007, p.1)

“One noteworthy intersection between statistics and mathematics is probability, which plays a critical role in statistical reasoning, but is also worthy of study in its own right as a subfield of mathematics.” (ASA, 2007, p.1) Candidates should develop an understanding and facility with probability concepts in service of statistical thinking.

To be prepared to develop student mathematical proficiency, all secondary mathematics teachers should know the following topics related to statistics and probability based on the Essential Concepts from *Catalyzing Change in High School Mathematics* (NCTM, 2018):

Essential Concepts in Quantitative Literacy

- Applying statistical reasoning with contextual data to evaluate risks or conclusions.

Essential Concepts in Visualizing and Summarizing Data

- Analyze contexts to identify appropriate data types and distributions with attention to describing the context (the shape, with appropriate measures of center and variability, including standard deviation or range) to compare two or more subgroups with respect to a variable.
- Data-analysis techniques and representations can be used to develop models of contextual situations and to generate and evaluate possible solutions to real problems involving those contexts.

Essential Concepts in Statistical Inference

- Understanding that the scope and validity of statistical inferences are dependent on the role of randomization in the study design including the differences in randomly selecting samples and randomly assigning subjects to experimental treatment groups.
- Understanding the impact of bias, such as sampling, response, or nonresponse bias, yielding results that are not representative of the population of interest.

Essential Concepts in Probability

- Two events are independent if the occurrence of one event does not affect the probability of the other event. Determining whether two events are independent can be used for finding and understanding probabilities.

- Conditional probabilities—that is, those probabilities that are “conditioned” by some known information—can be computed from data organized in contingency tables. Conditions or assumptions may affect the computation of a probability.

1e) Essential Concepts in Geometry, Trigonometry, and Measurement

Candidates demonstrate and apply understandings of major mathematics concepts, procedures, knowledge, and applications of geometry including using visual representations for numerical functions and relations, data and statistics, and networks, to provide a lens for solving problems in the physical world. *Essential Concepts in Geometry, Trigonometry, and Measurement include transformations; geometric arguments; reasoning and proof; applied problems; and non-Euclidean geometries.*

<p>Level 1 The Beginning Candidate</p>	<p>Level 2 The Developing Candidate</p>	<p>Level 3 The Competent Candidate</p>	<p>Level 4 The Accomplished Candidate</p>
<p>Candidate does not demonstrate conceptual understanding, procedural fluency, and/or factual knowledge for major mathematical concepts in Geometry, Trigonometry, and Measurement.</p>	<p>Candidate demonstrates conceptual understanding, procedural fluency, and/or factual knowledge, but is not able to apply the major mathematical concepts in Geometry, Trigonometry, and Measurement.</p>	<p>Candidate demonstrates and applies understandings of major mathematics concepts, procedures, knowledge, and applications of algebra and functions.</p> <p>Candidate uses technology to enhance their learning of Geometry, Trigonometry, and Measurement.</p>	<p>Candidate demonstrates and applies conceptual understanding, procedural fluency, and factual knowledge of major mathematical concepts in Number.</p> <p>Candidate uses technology to enhance their learning of Geometry, Trigonometry, and Measurement.</p> <p>Candidate makes connections within and among mathematical domains.</p>

Supporting Explanation:

Geometry is the mathematical study of shape, size, relative position of figures, and the properties of space and measurement is the quantification of these properties. Geometry and measurement provide another perspective for connecting mathematics to our physical world. Through this study candidates focus upon measurement, transformations, geometric arguments, reasoning and proof, and solving applied problems in the physical world.

“Measurement--quantifying size and shape-- is a basic tool for understanding and appreciating the world” (NCTM, 2014, p. 68) Candidates understand and distinguish among the ways in which unknown measurements can be determined and analyzed through the use of mathematical relationships coupled with known attributes described by quantitative measures for two and three dimensions.

“Geometric transformations are of various types and are distinguished by the geometric attributes that they preserve” (NCTM, 2014, p. 69). By approaching geometry through transformations, direct connections to algebra are accessible where often the subjects are seen as disconnected. Transformations highlight the need for a deep understanding of the concepts of congruence and similarity which are foundational to mathematical understanding.

The development of mathematical thinking from exploration and argument through reasoning to proof provides the framework for discourse and discussion and the sense making needed to create individual and collective understanding. There is a distinct difference in creating a proof as opposed to reciting a proof. Creation involves exploration, discovery, conjecture, and certification of reasoning.

As with other branches of mathematics, much of geometry can be used to describe, explore and make conclusions about real-world objects, events and activities. Candidates must experience and grapple with how geometry can describe and interpret problems from various circumstances in the real world.

To be prepared to develop student mathematical proficiency, all secondary mathematics teachers should know the following topics related to geometry and measurement based on the Essential Concepts from *Catalyzing Change in High School Mathematics* (NCTM, 2018):

Essential Concepts in Measurement

- Calculating or approximating measurements, including areas and volumes, should support an understanding of measurement and the objects being acted upon.
- Applying proportional relationships allows for the generation and comparison of measures, including length, area, or volume, between a known object and similar objects.

Essential Concepts in Transformations

- Applying geometric transformations to figures provides opportunities for describing the attributes of the figures preserved by the transformation and for describing symmetries by examining when a figure can be mapped onto itself.

- Transformations in geometry serve as a connection with algebra, both through the concept of functions and through the analysis of graphs of functions as geometric figures.

Essential Concepts in Geometric Arguments, Reasoning, and Proof

- Proofs of theorems can sometimes be made with transformations, coordinates, or algebra; all approaches can be useful, and in some cases one may provide a more accessible or understandable argument than another.

Essential Concepts in Solving Applied Problems and Modeling in Geometry

- Recognizing congruence, similarity, symmetry, measurement opportunities, and other geometric ideas, including right triangle trigonometry in real-world contexts, provides a means of building understanding of these concepts and is a powerful tool for solving problems related to the physical world in which we live.

Standard 2 Knowing and Using Mathematical Processes

Candidates demonstrate, within or across mathematical domains, their knowledge of and ability to apply the mathematical processes of problem solving; reason and communicate mathematically; and engage in mathematical modeling. Candidates apply technology appropriately within these mathematical processes.

Supporting Explanation:

“A worthy goal of mathematics instruction for any undergraduate is to develop not only knowledge of content but also the ability to work in ways characteristic of the discipline” (CBMS, 2012, p.5). While many categorize mathematics strictly by the content domains, it is critical for candidates to engage in, develop and demonstrate the mathematical processes which represent the mathematical habits of mind needed to learn and connect mathematics more broadly. As candidates expand their mathematical knowledge throughout their career, conceptual understanding and procedural fluency come from the ability to solve problems, reason through ideas, communicate understanding and use mathematical modeling to apply mathematics to real-world contexts. The development of these processes should occur as new knowledge is formed within and across mathematical domains. Candidates self-monitor and reflect upon their own cognitive experiences in developing their ability to use these processes. AMTE characterizes these processes as “doing mathematics as a sense-making activity that promotes perseverance, problem posing, and problem solving” (AMTE, 2017, Indicator C.1.2). Candidates develop these processes through doing mathematics, not be passively listening to how others have previously made sense of mathematics. Active, ongoing engagement in learning includes explaining, experimenting, and weighing their own understanding against the understanding of others.

Candidates should possess the ability to use mathematical and statistical processes and practices (NCTM, 2000; NGA & CCSSO, 2010; Shaughnessy, Chance, & Kranendonk, 2009) to solve problems; to use mathematical language with care and precision; and to explain their

mathematical thinking and critique the reasoning of others, using grade-appropriate definitions and interpretations for key mathematical concepts. Candidates can use mathematical modeling process to develop understandings of real-world situations through mathematics. Candidates exemplify the mathematical thinking that will be expected of their students. (AMTE, 2017, Indicator C.2.1)

Selecting Evidence for Standard 2:

Evidence for Standard 2 should reflect demonstration of the components by the candidates that reflect the candidates own experiences in using the mathematical processes to expand their own knowledge of the mathematical domains.

Many programs use a detailed analysis of course grades or transcript analysis (for post-baccalaureate programs) to provide evidence of candidates' demonstration of content and mathematical processes. Guidelines for the use of course grades are outlined on the NCTM website. Content-based assessment such as projects, course portfolio, or other course products aligned to mathematical domain competencies of *NCTM CAEP Mathematics Content for Secondary* accompanied by candidate performance data can also provide data for Standards 1 and 2.

Other possible assessments could include specific course-based assessments, portfolios, publications or presentations address the components of Standard 2.

2a) Problem Solving:

Candidates demonstrate a range of mathematical problem solving strategies to make sense of and solve non-routine problems (both contextual and non-contextual) across mathematical domains.

Level 1 The Beginning Candidate	Level 2 The Developing Candidate	Level 3 The Competent Candidate	Level 4 The Accomplished Candidate
Candidate solves non-routine problems (contextual or non-contextual) when given a strategy.	Candidate solves non-routine problems (contextual and non-contextual) when given a strategy.	Candidate demonstrates use of mathematical problem solving strategies to make sense of and solve contextual and non-contextual problems in more than one mathematical domain.	Candidate demonstrates coordination and unprompted use of multiple mathematical problem solving strategies when making sense of and solving contextual and non-contextual problems across mathematical domains. Candidate can compare strategies and make connections across domains.

Supporting Explanation:

Problem solving means engaging in a task for which the solution method is not known in advance. (NCTM, 2000, p. 52) Problem solving “is not only a goal of learning mathematics but also a major means of doing so. It is an integral part of mathematics, not an isolated piece of the mathematics program. [Candidates] require frequent opportunities to formulate, grapple with, and solve complex problems that involve a significant amount of effort. They are to be encouraged to reflect on their thinking during the problem-solving process so that they can apply and adapt the strategies (i.e. make a table, look for a pattern, work backwards, solve a simpler, etc.) they develop to other problems and in other contexts. By solving mathematical problems, [candidates] acquire ways of thinking, habits of persistence and curiosity, and confidence in unfamiliar situations that serve them well outside the mathematics classroom.” (NCTM, 2000b)

2b) Reasoning and Communicating:

Candidates organize their mathematical reasoning and use the language of mathematics to express their mathematical reasoning precisely, both orally and in writing to multiple audiences.

Level 1 The Beginning Candidate	Level 2 The Developing Candidate	Level 3 The Competent Candidate	Level 4 The Accomplished Candidate
Candidate is unable to organize their own mathematical reasoning and does not use the language of mathematics.	Candidate is able to organize their own mathematical reasoning using the language of mathematics with prompting and support. Candidates are able to express their mathematical reasoning orally or in writing.	Candidate is able to organize their own mathematical reasoning and use the language of mathematics to express their mathematical reasoning precisely, both orally and in writing to multiple audiences.	Candidate is able to organize their own mathematical reasoning and use of the language of mathematics to express their mathematical reasoning precisely, both orally and in writing to multiple audiences. Candidate seeks out opportunities to share their mathematical reasoning with professors, peers, and colleagues.

Supporting Explanation:

Conceptual understanding can only be documented through communication regarding the reasoning that a candidate uses to create that understanding. Candidates' coursework should include consistent rich opportunities to monitor and describe their mathematical reasoning, to convince themselves and others as to how they make conjectures, explore other lines of reasoning and come to conclusions in the different mathematical domains.

“Mathematical knowledge often grows in a cycle of inquiry and justification. Exploration, discovery and conjecture are aspects of mathematical reasoning that must be developed, including valuing and building upon the reasoning of others who are learning at the same time. Candidates must experience regular opportunities and circumstances in which they are required to reason and communicate their reasoning to others” (NCTM, 2018, p.39) and consider and critique the mathematical reasoning of others (NCTM, 2014, p. 17-20).

Mathematical communication is a way of sharing ideas and clarifying understanding. Through communication, ideas become objects of reflection, refinement, discussion, and amendment. When students are challenged to communicate the results of their thinking to others orally or in writing, they learn to be clear, convincing, and precise in their use of mathematical language. Explanations should include mathematical arguments and rationales, not just procedural descriptions or summaries. Listening to others' explanations gives students opportunities to develop their own understandings. Conversations in which mathematical ideas are explored from multiple perspectives help the participants sharpen their thinking and make connections (NCTM, 2000b).

2c) Mathematical Modeling and Use of Mathematical Models:

Candidates understand the difference between the mathematical modeling process and models in mathematics. Candidates engage in the mathematical modeling process and demonstrate their ability to model mathematics.

Level 1 The Beginning Candidate	Level 2 The Developing Candidate	Level 3 The Competent Candidate	Level 4 The Accomplished Candidate
Candidate does not demonstrate the ability to use the process of mathematical modeling or is unable to formulate and interpret mathematical models.	Candidate uses the process of mathematical modeling and formulates and represents, but needs assistance in analyzing and interpreting models.	Candidate uses the process of mathematical modeling to formulate, represent, analyze, and interpret mathematical models using a variety of tools including technology from real-world contexts or mathematical problems. Candidate can articulate the difference between a mathematical model and the mathematical modeling process.	Candidate uses the process of mathematical modeling to formulate, represent, analyze and interpret mathematical models derived from real-world context and mathematical problems. The candidate seeks opportunities to extend and reformulate models based upon analysis. Candidate can demonstrate the mathematical modeling process.

Supporting Explanation:

“Mathematical modeling is a process that uses mathematics to represent, analyze, make predictions or otherwise provide insight into real-world phenomena.” (ASA, 2007, p.7)

The word model has multiple meanings and uses in mathematics and mathematics education. It is important to distinguish between a “model” and the process of “mathematical modeling.” Students may model mathematics through the use of manipulatives, demonstration, acting out, diagrams, equations, and conceptual models of mathematics. This is in contrast to mathematical modeling which “uses mathematics to answer big, messy, reality-based questions.” (ASA, 2007, p.7)

The mathematical modeling cycle, as described in *Catalyzing Change*, begins with a real problem and involves multiple steps and often multiple iterations.

- Formulating the problem or questions
- Stating assumptions (often requiring simplifications of the real situation) and defining variables
- Restating the problem or question mathematically
- Solving the problem in the mathematical model
- Analyzing and assessing the solution and the mathematical model
- Refining the model, going back to the first steps if necessary
- Reporting the results. (NCTM, 2018, p.40)

Participation in modeling also facilitates the development of problem solving and communication. “Productive struggle is essential to the process.” (NCTM, 2018, p.40).

The use of mathematical models in learning mathematics can include but is not limited to

- Concrete Manipulatives
- Diagrams and graphing
- Equations
- Various technological tools such as spreadsheets, graphing, simulations, and virtual manipulatives

Standard 3: Knowing Students and Planning for Mathematical Learning

Candidates use knowledge of students and mathematics to plan rigorous and engaging mathematics instruction supporting students’ access and learning. The mathematics instruction developed provides equitable, culturally-responsive opportunities for all students to learn and apply mathematics concepts, skills, and practices.

Supporting Explanation:

As candidates begin to transition in their roles from learners of mathematics to those who will guide the learning of others, they move beyond their own learning to consider the diversity of individual students and groups of students that they will serve. They begin the process of understanding how the diversity, strengths, and identities of students influence how students

learn and consider these factors in planning for access and support in mathematics lessons. Their knowledge of students and mathematics lay the foundation for planning effective lessons that rely on a core set of pedagogical practices.

Planning is complex for candidates; it is anchored on their knowledge of pedagogical practices and their ability to meld these practices with their knowledge of students and mathematics. Knowledge of students' prior experience with mathematics, their language, culture, and interests as well as personal and mathematical strengths must be acquired and then attended to when selecting tasks, planning classroom interactions, and identifying supports. Knowing students, as previously described, is vital when planning for equitable teaching strategies.

Candidates must begin to anticipate how students will make sense of mathematics based upon the students' understandings, previous experience, interactions with each other, and beliefs about how they can learn mathematics. Candidates must also learn to develop accommodations for students with disabilities, and plan collaboratively for co-teaching with special educators to ensure that adaptations develop mathematical understanding and reasoning habits for the range of students in the classroom (Dieker et.al, 2011, pp 52-53).

Table 3.1
Adaptations to develop reasoning habits in the classroom

NCTM recommendation	Adaptations (by type of disability or deficit)					
	Visual	Auditory	Physical	Verbal	Behavioral	Memory/processing
Provide tasks that require students to figure things out for themselves.	Ensure that tasks have written and auditory components.	Ensure that visual images are part of the task.	Ensure that materials used (e.g., larger blocks) and space for work are accessible and easy to handle.	Repeat procedures; ask students in the groups to check understanding of peers.	Establish a positive peer group to discuss task. Foster positive reinforcement for on-task behavior.	Give concrete examples of steps within the task to help the student put it all together.
Ask students to restate the problem in their own words, including any assumptions they have made.	No adaptations needed.	Have students write their statements as well as give them orally.	Ensure that the student can see/hear as others share their thoughts.	Use assistive technology to share student's thoughts in text-to-speech format.	Ensure that student has processing time and will be able to share thoughts in a positive climate.	Allow student to write down response before sharing.
Give students time to analyze a problem intuitively, explore the problem further by using models, and then proceed to a more formal approach.	Remember that a student processing through only one modality may need more time; ensure that models are presented physically or orally—if possible, furnish 3D models for tactile input.	Remember that a student processing through only one modality may need more time; ensure that models are presented physically or orally and with written phrases.	Ensure that all workspace is accessible and that students can access models.	No adaptations needed.	Use a timer so that students have a concept of how much time they have to complete the task; list the steps of what they are to do within each allotted block of time.	Clarify any language that may delay processing and ensure that the student can make a basic statement about the problem before moving forward.
Resist the urge to tell students how to solve a problem when they become frustrated; find other ways to support.	Ensure that you present enough verbal information for understanding.	Ensure that you present enough visual information for understanding.	Ensure that physical limitations do not impair ability to understand.	Ensure that this student has a way to share his or her thoughts through pictures (e.g., Google images) or through text-to-speech technology.	Monitor frustration levels to ensure success and avoid a behavioral outburst. Furnish scaffolding to ensure moving forward and to avoid frustration.	Check that the language or the processing speed of dialogue is not impairing students' ability to understand.
Ask students questions that will prompt their thinking—for example, "Why does this work?" or "How do you know?"	Stand near or give student a cue for a question addressed to him or her.	Offer both oral and written summaries of questions (e.g., overhead, whiteboard); use another student to make visual images of questions on the board.	Allow ways to respond besides raising hand (e.g., have student with physical disability move forward).	Allow students other ways to answer questions (e.g., whiteboard, text-to-speech format, images).	To decrease anxiety, prepare student for direct questions; ensure that the student gets positive reinforcement from you and peers throughout questioning.	Use cues such as standing near a student before asking a question or supply two questions the student can answer ahead of time.

Table 3.1—Continued
Adaptations to develop reasoning habits in the classroom

NCTM recommendation	Adaptations (by type of disability or deficit)					
	Visual	Auditory	Physical	Verbal	Behavioral	Memory/processing
Allow adequate wait time after a question for students to formulate their own reasoning.	Ensure that student is seated where he or she can hear the question clearly; repeat student questions for the class.	Ensure time for student to process the written question or allow time for the sign language interpreter.	No adaptations needed.	Ensure that student has a way to share thoughts or to clarify if he or she does not understand.	Ensure that student is attending to you as you pose the question; ask student to repeat the question if you feel that he or she was not focused.	Use wait-time research; if student has a language-processing issue, consider doubling wait time.
Encourage students to ask probing questions to themselves and one another.	Ensure adequate physical representation of concept if questions are related to a visual image.	Allow students to use assistive devices or interpreter for sharing questions.	No adaptations needed.	Allow student to use assistive devices or interpreter for sharing questions.	Offer examples of appropriate questions and ensure positive peer interactions.	Help students word questions or to process information as needed.
Expect students to communicate their reasoning to their classmates and the teacher orally and in writing by using proper mathematical vocabulary.	Establish rules for turn taking for a student who cannot see hands raised, body language, and the like. Ensure that written questions are in an accessible format.	Educate class on students' use of an interpreter or lip reading to ensure equal participation.	Supply accessible table and materials in the group setting; ensure use of appropriate assistive devices for written work.	Furnish a way for students who have limited speech to share questions (e.g., voice to text) and educate peers on how to work effectively with such students.	Offer a safe way for student to participate and minimize any possibility of negative peer feedback or potential anger through the reasoning process.	Ensure that the student understands key vocabulary words to participate in a group setting; assist with scribing responses if processing language is difficult.
Highlight exemplary explanations and have students reflect on what makes them effective.	Ensure that all students are celebrated for how they learn and contribute to the classroom.					
Establish a classroom climate in which students feel comfortable sharing their mathematical arguments and productively critiquing the arguments of others.	Establish rules for sharing that include oral cues.	Establish rules for sharing that include visual cues.	Ensure that the climate allows those with physical limitations to participate.	Ensure that students can be engaged in ways beyond oral responses (e.g., by writing on dry-erase boards).	Discuss and role-play how to give and receive feedback to arguments. Have daily social goals grounded in this area for all students.	Ensure that the climate allows time to process questions, contribute to group discussions, and write responses.

Candidates know students and can identify the supports students need to engage in collaboration, discourse, and productive struggle. When teaching emergent bilinguals or multilinguals, traditionally identified as English Language Learners, candidates are aware of the importance of establishing a linguistically sensitive social environment that supports learning English while learning mathematics. Candidates plan for the use of mathematics tools and modeling as a resource, and recognize cultural and linguistic differences as intellectual resources. Candidates are aware of the language demands facing emergent bilinguals or multilinguals as listening, speaking, reading, and writing are required modes of communication in today's mathematics classrooms (Celedón-Pattichis and Ramirez, 2012). No more can the statement that mathematics is a universal language be employed when teaching emergent bilinguals as today's students are required to consistently communicate their understanding and reasoning.

Guiding Principles for Teaching Mathematics to English Language Learners

1. Challenging mathematical tasks: Students at all levels of English language development need challenging mathematical tasks, made accessible through supports that clarify their understanding of the task. Although the tasks may be the same for all levels, the teacher actions required for students to have access to them and to communicate their understanding often differ at each level.
2. Linguistically sensitive social environment: Mathematical learning occurs in a linguistically sensitive social environment that takes into consideration linguistic demands and discourse elements (Chval and Chávez 2011/2012; Chval and Khisty 2009) and is characterized by teacher-supported, ongoing, high-quality interactions that include all forms of communication between teachers and students and between students and students.
3. Support for learning English while learning mathematics: Facility with the English language is acquired when ELLs learn mathematics through effective instructional practices, including support structures that scaffold students' language development, engage students in MDCs, make mathematics content linguistically comprehensible to them, and assess their progress in reaching predetermined linguistic and mathematical goals.
4. Mathematical tools and modeling as resources: Mathematical tools and mathematical modeling provide a resource for ELLs to engage in mathematics and communicate their mathematical understanding and are essential in developing a community that enhances discourse.
5. Cultural and linguistic differences as intellectual resources: Students' cultural and linguistic differences in the mathematics community should be viewed as intellectual resources rather than as deficits and should be used in the classroom to connect to prior knowledge and to create a community whose members value one another's ways of engaging in mathematics.

Planning articulates and supports tasks, mathematical language acquisition and use, contexts, and learning trajectories that enable and motivate students to participate meaningfully and make sense of the mathematics they are exploring and learning. Planning includes learning and using connections that exist among standards, curriculum documents, instructional materials, and assessment frameworks. Candidates begin to use their knowledge of students, mathematics, and pedagogy to identify, access, assess and use supplemental resources.

Well prepared beginning teachers of mathematics apply knowledge of a full range of students, standards, pedagogical content knowledge, knowledge of curriculum and effective and equitable

teaching practices to support students understanding, elicit and use evidence of students' thinking. A full range of students means all subgroups of students based on the CAEP definition: group differences (e.g., race, ethnicity, ability, gender identity, gender expression, sexual orientation, nationality, language, religion, political affiliation, and socio-economic background) (CCSSO, 2013, p. 21).

Selecting Evidence for Standard 3:

Evidence for Standard 3 may be demonstrated through assessments of planning (lesson/unit plans), observations of teaching (student teaching, internship, practicum, etc.), or other assessments that particularly address how knowledge of students was used when instructional choices were made.

The table below outlines the connections between the components in Standard 3 and the AMTE and NCTM descriptions of equitable and effective teaching.

Connecting Components and Indicators on Teaching Meaningful Mathematics

SPA Component	AMTE Standards for the Preparation of Teachers of Mathematics	NCTM Principles to Actions Mathematics Teaching Practices and Principles
<p>3a) Student Diversity: Candidates identify and use students’ individual and group differences to plan rigorous and engaging mathematics instruction that supports students’ meaningful participation and learning.</p>	<p>C.1.5. Analyze Mathematical Thinking: Well-prepared beginning teachers of mathematics analyze different approaches to mathematical work and respond appropriately.</p> <p>C.4.1. Provide Access and Advancement: Well-prepared beginning teachers of mathematics recognize the difference between access to and advancement in mathematics learning and work to provide access and advancement for every student. NOT sure of this one</p> <p>C.3.1. Anticipate and Attend to Students’ Thinking About Mathematics Content: Well-prepared beginning teachers of mathematics anticipate and attend to students’ mathematical thinking and mathematical learning progressions.</p> <p>C.3.2. Understand and Recognize Students’ Engagement in Mathematical Practices: Well-prepared beginning teachers of mathematics understand and recognize mathematical practices within what students</p>	<p>Access and Equity Principle An excellent mathematics program requires that all students have access to a high-quality mathematics curriculum, effective teaching and learning, high expectations, and the support and resources needed to maximize their learning potential.</p> <p>Effective Mathematics Teaching Practices: Establish mathematics goals to focus learning. Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses the goals to guide instructional decisions.</p> <p>Use and connect mathematical representations. Effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding</p>

	<p>say and do across many mathematical content domains, with in-depth knowledge of how students use mathematical practices in particular content domains.</p>	<p>of mathematics concepts and procedures and as tools for problem solving.</p> <p>Implement tasks that promote reasoning and problem solving. Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies.</p> <p>Support productive struggle in learning mathematics. Effective teaching of mathematics consistently provides students, individually and collectively, with opportunities and</p>
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<p>3b) Students' Mathematical Strengths: Candidates identify and use students' mathematical strengths to plan rigorous and engaging mathematics instruction that supports students' meaningful participation and learning.</p>	<p>C.1.6. Use Mathematical Tools and Technology: Well-prepared beginning teachers of mathematics are proficient with tools and technology designed to support mathematical reasoning and sense making, both in doing mathematics themselves and in supporting student learning of mathematics.</p> <p>C.2.1. Promote Equitable Teaching: Well-prepared beginning teachers of mathematics structure learning opportunities and use teaching practices that provide access, support, and challenge in learning rigorous mathematics to advance the learning of every student.</p> <p>C.2.2. Plan for Effective Instruction: Well-prepared beginning teachers of mathematics attend to a multitude of factors to design mathematical learning opportunities for students, including content, students' learning needs, students' strengths, task selection, and the results of formative and summative assessments.</p> <p>C.4.3. Draw on Students' Mathematical Strengths: Well-prepared beginning teachers of mathematics identify and implement practices that draw on students' mathematical, cultural, and linguistic resources/strengths and challenge policies and practices grounded in deficit- based thinking</p>	<p>supports to engage in productive struggle as they grapple with mathematical ideas and relationships.</p> <p>Pose purposeful questions. Effective teaching of mathematics uses purposeful questions to assess and advance students' reasoning and sense making about important mathematical ideas and relationships.</p> <p>Build procedural fluency from conceptual understanding. Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems.</p>
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<p>3c) Students' Mathematical Identities: Candidates understand that teachers' interactions impact individual students by influencing and reinforcing students' mathematical identities, positive or negative, and plan experiences and instruction to develop and foster students' positive mathematical identities.</p>	<p>C.3.3. Anticipate and Attend to Students' Mathematical Dispositions: Well-prepared beginning teachers of mathematics know key facets of students' mathematical dispositions and are sensitized to the ways in which dispositions may affect students' engagement in mathematics.</p>	
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3a) Student Diversity:

Candidates identify and use students' individual and group differences when planning rigorous and engaging mathematics instruction that supports students' meaningful participation and learning.

Level 1 The Beginning Candidate	Level 2 The Developing Candidate	Level 3 The Competent Candidate	Level 4 The Accomplished Candidate
Candidate does not use students' individual differences or group differences in planning rigorous and engaging mathematics instruction.	Candidate uses students' individual or group differences in planning rigorous and engaging mathematics instruction for a subset of students.	Candidate uses students' individual and group differences in planning rigorous and engaging mathematics instruction that supports meaningful participation and learning by across a full range of students.	Candidate uses students' individual and group differences in planning rigorous and engaging mathematics instruction that supports meaningful participation and learning by each and every student.

Supporting Explanation:

Candidates must work to understand the critical and deliberate professional learning necessary to acknowledge and build upon the diversity of individual students and groups of students, particularly those whose learning experiences and needs are different from their own. For the purposes of these standards, NCTM builds upon the CAEP definition of diversity: (1) Individual differences (e.g., personality, interests, and life experiences), and (2) group differences (e.g., race, ethnicity, ability, gender identity, gender expression, sexual orientation, nationality, language, religion, political affiliation, and socio-economic background) (CCSSO 2013, p. 21).

It is a programmatic responsibility to ensure each candidate attends specifically to a full range of students, with particular attention to students who are traditionally underrepresented and/or underserved in the learning of mathematics. Additionally, candidates must ensure that students with exceptionalities receive the appropriate accommodations. Candidates recognize the need to learn about students' mathematical experiences, community, language and culture to develop relationships and plan instruction that attends to, honors and celebrates the diversity of the students while focusing on instruction that is supportive of all students regardless of their primary language or their exceptionalities. They plan instruction that supports students' participation in making sense of important mathematical concepts that include opportunities to develop an understanding of mathematics by speaking, reading, writing and listening. They engage other educators in learning about student diversity and plan with this in mind.

Candidates learn about students by analyzing student reflections and student work, and through interactions, observations, surveys, student interviews, etc. Programs provide opportunities for candidates to use that knowledge to establish math goals within a learning progression, select tasks that promote reasoning and problem solving, design activities that support student engagement in rigorous mathematics while making accommodations for groups and individual students, and plan whole group and small group instruction grouping and regrouping students as needed. Candidates build on students' current understanding to develop procedural fluency from conceptual understanding, consider representations, tools and technology that could be used in the classroom as well as consider what students will choose to use or how they might solve particular problems. Candidates identify questions related to the mathematical goals that will foster classroom discourse and support the development of understanding and fluency with mathematics and mathematical language.

3b) Students Mathematical Strengths:

Candidates identify and use students' mathematical strengths to plan rigorous and engaging mathematics instruction that supports students' meaningful participation and learning.

Level 1 The Beginning Candidate	Level 2 The Developing Candidate	Level 3 The Competent Candidate	Level 4 The Accomplished Candidate
Candidate does not use students' mathematical strengths in planning rigorous and engaging mathematics instruction.	Candidate uses students' mathematical strengths in planning rigorous and engaging mathematics instruction for a subset of students.	Candidate uses students' mathematical strengths in planning rigorous and engaging mathematics instruction that supports meaningful participation and learning by across a full range of students.	Candidate uses students' mathematical strengths in planning rigorous and engaging mathematics instruction that supports meaningful participation and learning by each and every student.

Supporting Explanation:

As candidates plan, they build on what they know about students and the mathematics that students know. They choose classroom resources and contexts based on their understanding of mathematical learning progressions and what understandings and skills their students possess within the progressions. Candidates select tasks that are deliberately chosen to make mathematics accessible to students and plan additional mathematics experiences that are designed to lead to rigorous learning experiences.

Candidates understand the unique learning needs of their students and plan learning experiences to meet those needs. They plan instruction that capitalizes on student culture, interest, and language, available technology, manipulatives, and other tools. They design learning experiences that engage students and allow them to interact with the mathematics beyond superficial production of solutions. Their plans are indicative of their understanding of research related to building on the strengths of all students, especially those with disabilities. These approaches include but are not limited to using inquiry instruction, visual enhancements, and problem-solving schemas (Dieker et. al., 2011, p. 64-68). Candidates know what mathematics students know and do not assume that when a student does not know English it does not indicate that they do not know mathematics. Candidates are able to communicate the rationale for their instruction decisions and how those decisions lead to student engagement and learning.

Candidates use their conceptual understanding of mathematics, and knowledge of tools, technology, and mathematical rigor to plan appropriate instruction. They plan building on what students know and adjust instruction to accommodate the full range of learning needs within their classes. They plan for students' use of tools and technology including manipulatives, drawings, handheld technology, presentation/interactive technologies, and mathematics-specific technology. They understand that multimodal instruction supports a full range of students in reasoning, problem solving, development of conceptual understanding, and communicating about rigorous mathematics. Candidates take into account their knowledge of students' needs and strengths as they anticipate student responses and interactions during classroom discourse and with the mathematics content. They demonstrate insights into strengths and limitations of various tools and technologies as they plan for instruction.

3c) Positive Mathematical Identities:

Candidates understand that teachers’ interactions impact individual students by influencing and reinforcing students’ mathematical identities, positive or negative, and plan experiences and instruction to develop and foster positive mathematical identities.

<p>Level 1 The Beginning Candidate</p>	<p>Level 2 The Developing Candidate</p>	<p>Level 3 The Competent Candidate</p>	<p>Level 4 The Accomplished Candidate</p>
<p>Candidate does not recognize that teachers’ interactions impact individual students by influencing and reinforcing student’s mathematical identities, positive or negative; or candidate does not plan experiences and instruction to develop and foster students’ positive mathematical identities for a subset of students.</p>	<p>Candidate understands that teachers’ interactions impact individual students by influencing and reinforcing student’s mathematical identities, positive or negative.</p> <p>Candidate plans experiences and instruction to develop and foster students’ positive mathematical identities for a subset of students.</p>	<p>Candidate understands that teachers’ interactions impact individual students by influencing and reinforcing student’s mathematical identities, positive or negative.</p> <p>Candidate plans experiences and instruction to develop and foster students’ positive mathematical identities across a full range of students.</p>	<p>Candidate understands that teachers’ interactions impact individual students by influencing and reinforcing student’s mathematical identities, positive or negative.</p> <p>Candidate plans experiences and instruction to develop and foster students’ positive mathematical identities for each and every student.</p>

Supporting Explanation:

Candidates develop an understanding of their own mathematical identity and understand the importance of learning about students' mathematical identities. Mathematical identities are “the dispositions and deeply held beliefs that students develop about their ability to participate and perform effectively in mathematical contexts and to use mathematics in powerful ways across the contexts of their lives” (Aguirre, 2013, p. 14). They refer to “how students see themselves and how they are seen by others, including teachers, parents and peers as doers of mathematics” (Aguirre, 2013, p. 13).

Candidates are aware of the impact of their verbal and non-verbal interactions with students. They realize that their tone, expressions, words and actions influence students' mathematical identities and beliefs, and they strive to promote positive student mathematical identities. Candidates demonstrate this awareness of their **influence** and ability to make a positive impact through their planning and teaching during field placements, student teaching, or other practicum experiences.

Programs ensure that candidates understand that all mathematics teachers are identity workers in that they contribute to the kinds of identities students develop both inside and outside the classroom (Gutiérrez, 2013). Students, as well as adults, harbor perceptions about what someone who is good at mathematics “looks like” more so than for most subjects; even very young students can identify who in their classrooms are “good” at mathematics, often choosing those who are quick to recall facts or perform algorithms. Well-prepared candidates know that research and standards provide a different description of what being “good at mathematics” entails. For example, *Adding it Up* (NRC, 2001) described a productive disposition as “the inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one’s own efficacy” (p. 116). Well-prepared candidates seek to actively position all learners as mathematical doers. They understand that developing positive mathematical identities begins with focusing on robust goals for what is important to know and be able to do in mathematics and includes doing mathematics for one’s own sake, not just to score well on mathematics tests.

Candidates analyze their task selections and implementation, reflecting on the ways they may shape students' mathematical identities. In addition to considering the extent to which the mathematics of the task positions learners as doers, well-prepared beginners consider the contexts of tasks. Some contexts used, such as baseball, rocket launching, or farming, may privilege a particular group of students who are familiar and interested in that context. Some classroom practices, such as board races and timed tests, have long-standing history in U.S. classrooms, despite the fact that they exclude those who need more processing time while also communicating that those who are fast are good at mathematics.

Candidates attend to mathematical practices (NGA & CCSSO, 2010) such as reasoning, problem solving, persistence, modeling, and precision, as well as the effective mathematics teaching practices (NCTM, 2014) when they teach students in order to build positive mathematical identities for students. Candidates are aware of students' strengths, differences, and interests when planning and delivering instruction and build upon them to develop positive student mathematical identities.

Standard 4: Teaching Meaningful Mathematics

Candidates implement effective and equitable teaching practices to support rigorous mathematical learning for a full range of students. Candidates establish rigorous mathematics learning goals, engage students in high cognitive demand learning, use mathematics specific tools and representations, elicit and use student responses, develop conceptual understanding and procedural fluency, and pose purposeful questions to facilitate student discourse.

Supporting Explanation:

Teaching mathematics requires accessing various knowledge and skills. Specifically, effective and equitable teaching *uses* knowledge of mathematics content, mathematical processes, and knowledge of learners, each of which have been addressed in Standards 1-3. Note that Standard 3 also focuses on teaching as it relates to the learner. In Standard 4, the focus is more explicitly on effective teaching of mathematics, focusing on aspects central and specific to the content, such as using mathematical tools and representations and involving students in active engagement with meaningful mathematics.

Standard 4 focuses on effective *and* equitable teaching. These two ideas are implicit and explicit in each of the components described within the standards. Teaching mathematics is complex and incorporating effective and equitable teaching practices is a career-long endeavor. As noted in the *AMTE Standards for Preparing Teachers of Mathematics* (2017), equitable learning outcomes requires (a) clear and coherent mathematical goals for students' learning, (b) expectations for the collective work of students in the classroom, (c) effective methods of supporting the learning of mathematics by each student, and (d) provision of appropriate tools and resources targeted to students' specific needs. (Indicator C.2.1). The NCTM Mathematics Teaching Practices (NCTM, 2014) describe eight mathematics teaching practices that lead to equitable and effective teaching. The table below outlines the connections between the components in Standard 4 and the AMTE and NCTM descriptions of equitable and effective teaching.

Connecting Components and Indicators on Teaching Meaningful Mathematics

<p>SPA Component</p>	<p><i>AMTE Standards for the Preparation of Teachers of Mathematics</i></p>	<p><i>NCTM Principles to Actions Mathematics Teaching Practice</i></p>
<p>4a) Establish Rigorous Mathematics Learning Goals: Candidates establish rigorous mathematics learning goals for students situated within learning progressions, mathematics standards and practices, and the purposes for learning mathematics.</p>	<p>C.2.1. Promote Equitable Teaching. Well-prepared beginning teachers of mathematics structure learning opportunities and use teaching practices that provide access, support, and challenge in learning rigorous mathematics to advance the learning of every student.</p>	<p>Establish mathematics goals to focus learning. Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses the goals to guide instructional decisions.</p>
<p>4b) Engage Students in High Cognitive Demand Learning: Candidates analyze, modify, sequence, and implement tasks to engage students in high cognitive demand mathematical learning experiences that promote reasoning and sense making.</p>	<p>C.2.3. Implement Effective Instruction. Well-prepared beginning teachers of mathematics use a core set of pedagogical practices that are effective for developing students’</p>	<p>Implement tasks that promote reasoning and problem solving. Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies. Support productive struggle in learning mathematics. Effective teaching of mathematics consistently provides students, individually and collectively, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships.</p>

<p>4c) Incorporate Mathematics-Specific Tools: Candidates select mathematics-specific tools, including technology, to support students' learning, understanding, and application of mathematics and integrate tools into instruction.</p>	<p>meaningful learning of mathematics.</p>	<p>Use and connect mathematical representations. Effective teaching of mathematics engages students in making connections among mathematical representations to deepen understanding of mathematics concepts and procedures and as tools for problem solving.</p>
<p>4d) Use Mathematical Representations: Candidates select mathematical representations to support students' learning, understanding and application of mathematics and implement and connect representations during instruction.</p>		<p>Elicit and use evidence of student thinking. Effective teaching of mathematics uses evidence of student thinking to assess progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning.</p>
<p>4e) Elicit and Use Student Responses: Candidates use multiple student responses, potential challenges and misconceptions, and highlight students' thinking as a central aspect of mathematics teaching and learning.</p>		<p>Build procedural fluency from conceptual understanding. Effective teaching of mathematics builds fluency with procedures on a foundation of conceptual understanding so that students, over time, become skillful in using procedures flexibly as they solve contextual and mathematical problems.</p>
<p>4f) Develop Conceptual Understanding and Procedural Fluency: Candidates use conceptual understanding to build procedural fluency for students through instruction that includes explicit connections between concepts and procedures.</p>		

4g) Facilitate Discourse: Candidates pose purposeful questions to facilitate discourse among students that ensures that each student learns rigorous mathematics and builds a shared understanding of mathematical ideas.

Pose purposeful questions. Effective teaching of mathematics uses purposeful questions to assess and advance students' reasoning and sense making about important mathematical ideas and relationships.
Facilitate meaningful mathematical discourse. Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments.

Selecting Evidence for Standard 4:

Evidence provided for this standard must be *enacted* instruction, a unit plan is not sufficient, it must actually be implemented in a classroom. The components described in Standard 4 may not be evident in a single day of instruction, but must be evident within a unit of instruction. For example, procedural fluency may not be represented in a particular day in which a single strategy or task is developed, but a unit of study must make explicit connections between conceptual understanding and procedural fluency. Therefore, a unit plan that was implemented, along with lesson observations and reflections on teaching would combine to provide evidence across Standard 4 components. Evidence can also be from other teaching performances and reflections such as student interviews and transcript analysis or standards performance instruments such as the edTPA.

Definition of Terms:

Design: The term ‘design’ is defined in the Oxford dictionary to mean, “The art or action of conceiving of and producing a plan”. In these standards, we use the term design, rather than plan, to communicate the strategic and creative process of determining a plan, which then supports effective teaching.

Equitable teaching practices: Teaching practices that tend to the needs of each and every student. As described in the *AMTE Standards for Preparing Teachers of Mathematics* (2017), this requires structuring learning opportunities to provide access, support, and challenge in learning mathematics, which includes “considering students’ individual needs, cultural experiences, and interests as well as prior mathematical knowledge when selecting tasks and planning for mathematics instruction (Leonard, Brooks, Barnes-Johnson, & Berry, 2010).” (C.2.1).

Effective teaching practices: This phrase refers to a set of mathematics teaching practices that optimize learning for students. The set used in these standards are based on *Principles to Actions Mathematics Teaching Practices* (NCTM, 2014) (see Table above). “Effective teaching practices (e.g., engaging students with challenging tasks, discourse, and open-ended problem solving) have the potential to open up greater opportunities for higher-order thinking and for raising the mathematics achievement of all students, including poor and low-income students.” (NCTM, 2014, p. 63)

Effective teaching practices must attend to equitable teaching practices. *Catalyzing Change in High School Mathematics* (NCTM, 2018) offers a useful and important crosswalk of effective and equitable mathematics teaching practices.

<INSERT FIGURE 2 FROM CATALYZING CHANGE, if POSSIBLE>

Pedagogical Content Knowledge: A core part of content knowledge for teaching that includes: core activities of teaching, such as figuring out what students know; choosing and managing representations of ideas; appraising, selecting and modifying textbooks; deciding among

alternative courses of action and analyzing the subject matter knowledge and insight entailed in these activities. (CAEP, 2013)

The adjective *rigorous* is used throughout standards documents “rigorous mathematics”, “rigorous goals” “rigorous mathematics instruction” (e.g., Catalyzing Change, Principles to Actions, and the AMTE Standards for the Preparation of Teachers of Mathematics). The term rigor is described (though not defined) in the Key Shifts document related to the Common Core State Standards for Mathematics (NGA & CCSSO, 2010):

“Rigor refers to deep, authentic command of mathematical concepts, not making math harder or introducing topics at earlier grades. To help students meet the standards, educators will need to pursue, with equal intensity, three aspects of rigor in the major work of each grade: conceptual understanding, procedural skills and fluency, and application.” Based on this description of Rigor, we define the following terms to support the interpretation of this standard, its components, and related rubrics.

Rigorous Mathematics: Thorough and careful treatment of the mathematics that attends equitably to conceptual understanding, procedural skills and fluency, and application. This includes attention to important representations, tools, conceptual foundations, and connections to relevant contexts.

Rigorous Mathematics Learning Goals: Clear expectations for learners that focus on conceptual understanding, procedural skills and fluency, and application, including attention to the purposes for learning mathematics, content standards and process standards (See Standard 1 and Standard 2).

Note: Rigorous mathematics does *not* mean that the problems use difficult values making the problem more computationally difficult; rigorous mathematics does mean that students engage in productive struggle as they solve mathematical tasks.

4a) **Establish Rigorous Mathematics Learning Goals:**

Candidates establish rigorous mathematics learning goals for students based on mathematics standards and practices.

Level 1 The Beginning Candidate	Level 2 The Developing Candidate	Level 3 The Competent Candidate	Level 4 The Accomplished Candidate
Candidate establishes mathematics learning goals for students which lack rigor.	Candidate establishes mathematics learning goals for students which demonstrate some level of rigor but are not situated within learning progressions, mathematics standards and practices, or the purposes for learning mathematics.	Candidate establishes rigorous mathematics learning goals for students situated within learning progressions, mathematics standards and practices, and the purposes for learning mathematics.	Candidate establishes rigorous mathematics learning goals for students situated within learning progressions, mathematics standards and practices, and the purposes for learning mathematics and recognizes and uses connections when establishing goals.

Supporting Explanation:

Candidates establish rigorous mathematics learning goals for students situated within learning progressions, mathematics standards and practices, and the purposes for learning mathematics.

As prospective teachers begin to prepare for instruction, they learn that lessons and units require clear goals and/or objectives in order to set up effective instruction. As noted in *Principles to Actions*, “Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses goals to guide instructional decisions.” (NCTM, 2014, p.10, 12) In fact, the practice of establishing clear goals that indicate what mathematics students are learning provides the starting point and foundation for intentional and effective teaching (p. 14). And, such learning goals must attend to student needs, cultural experiences, and interests (AMTE, 2017).

Establishing learning goals goes beyond stating or writing goals, establishing is about the enactment of instruction, not the design of instruction. A teacher establishes learning goals through actions at the beginning of a lesson and throughout a lesson to ensure that students ultimately understand the purpose of the days’ tasks and related discussions. To be clear, the expectation for this component is *not* that the candidate states or writes the objective at the start of class. In some cases, this may be appropriate, but other times, this can take away from an important mathematical discovery later in the lesson. But it does mean that the candidate is able to communicate a purpose for each element of the lesson and that at the conclusion of the lesson, it is clear that they have established rigorous goals that were reinforced throughout the lesson. The establishment of clear goals not only guides teachers’ decision making during a lesson but also focuses students’ attention on monitoring their own progress toward the intended learning outcomes (NCTM, 2014).

The starting point for mathematics teaching is designing rigorous mathematics learning goals. This requires attention to three things, all of which must be incorporated into a unit of study, though all three may not be present in every lesson within a unit - learning progressions, learning goals, and purpose. While the lessons will all be developmentally sequenced, some lessons may attend to mathematical connections while others may focus on connecting to relevant contexts. Each of these three ideas are described here to clarify meaning.

Learning progressions describe how students make transitions from their prior knowledge to more sophisticated understandings. For some topics, there are well-articulated learning progressions, for others there are not. In either case, candidates must demonstrate that they can design a series of lessons in which ideas build upon previous learning prior to the unit and within the unit so that students see the connections among mathematical ideas. Learning progressions also identify intermediate understandings and link research on student learning to instruction. When rigorous mathematics learning goals are implemented, both the teacher candidate and their students are able to answer questions such as:

- What mathematics is being learned and why is it important?
- How does the topic relate to what has already been learned?

Second, rigorous mathematics learning goals attend to the purposes for learning mathematics and the content and processes (NCTM, 2018).

- 1) Mathematics content that comprehensively addresses and connects conceptual understanding, procedural knowledge, and applications;
- 2) Mathematics processes that are connected to the content (e.g., not only stating that students will “Reasons abstractly and quantitatively,” but describing more specifically such reasoning as it is connected to the content learning goals;

Within a unit and for every lesson, candidates should be able to articulate goals that address both content and processes.

Third, rigorous mathematics learning goals attend to purpose. Purposes for learning mathematics may include how it relates to other mathematical ideas, but must also attend to the relevance of the mathematics in general. Important to rigorous mathematics is that the mathematics is relevant to students’ lives.

4b) Engage Students in High Cognitive Demand Learning:

Candidates select or develop and implement high cognitive demand tasks to engage students in mathematical learning experiences that promote reasoning and sense making.

Level 1 The Beginning Candidate	Level 2 The Developing Candidate	Level 3 The Competent Candidate	Level 4 The Accomplished Candidate
Candidate selects tasks without regard to engaging students in high cognitive demand mathematical learning experiences.	Candidate selects or develops tasks which could engage students in high cognitive demand mathematical learning experiences, but implementation fails to maintain a high cognitive demand with students.	Candidate selects or develops and implements tasks to engage a full range of students in high cognitive demand mathematical learning experiences that promote reasoning and sense making.	Candidate analyzes, modifies, sequences, and implements tasks to engage each and every student in high cognitive demand mathematical learning experiences that promote reasoning and sense making.

Supporting Explanation:

A task is a planned learning activity for students. It may be a single problem or a set of problems. High cognitive demand tasks” require complex and non-algorithmic thinking, require students to explore and understand the nature of mathematical concepts, processes or relationships, require students to analyze the task...” (Smith & Stein, 1998, p. 348). High cognitive demand tasks are experiences that involve active exploration or that encourage students to use procedures in ways that are meaningfully connected with concepts or understanding (NCTM, 2014). High-level tasks require students to think abstractly and make connections to mathematical concepts. These tasks can use procedures, but in a way that builds connections to mathematical meanings and understandings. (Stein & Smith, 1998). Building connections is a defining feature of a high cognitive demand task.

“From the perspective of this taxonomy, mathematical tasks are viewed as placing *higher level* cognitive demands on students when they allow students to engage in active inquiry and exploration or encourage students to use procedures in ways that are meaningfully connected with concepts or understanding” (NCTM, 2014, p. 19)

When a lesson focuses on procedures, high cognitive demand tasks are ones that connect a procedure or set of procedures to develop a students’ deeper level of understanding of mathematical concepts and ideas, rather than solely focus on learning a procedure as the sole focus. These higher-level tasks require some degree of thinking; students cannot solve them by simply implementing a set of steps communicated by the teacher. The task must engage students with conceptual ideas, meaning the task triggers the procedure that is needed to complete the task, and develop understanding. In other words, high cognitive demand tasks require multifaceted thinking, and the exact plans to solve the task are not clearly proposed in the instructions.

High cognitive demand learning engages students in reasoning, problem solving, and modeling, and such tasks support equitable mathematics teaching. *Catalyzing Change in High School Mathematics* (NCTM, 2018) lists mathematics teaching practices (NCTM ,2014) and equitable teaching practices related to enacting tasks:

- Engage students in tasks that provide multiple pathways for success and that require reasoning, problem solving, and modeling, thus enhancing each student’s mathematical identity and sense of agency.
- Engage students in tasks that are culturally relevant.
- Engage students in tasks that allow them to draw on their funds of knowledge (i.e., the resources that students bring to the classroom, including their home, cultural, and language experiences). (NCTM, 2018, p. 32)

Competent and accomplished candidates are able to enact instruction in which students are reasoning, solving problems, and making connections within mathematics and connecting mathematics to meaningful contexts.

In order to analyze, modify and sequence high cognitive tasks, candidates must attend to a “multitude of factors to design mathematical learning opportunities for students, including content, students' learning needs, students’ strengths, task selection, and the results of formative

and summative assessments.” (AMTE, 2017, C.2.2). In the enactment of instruction, candidates use instructional strategies to ensure that every student is accountable to reason and make connections. An important distinction between the competent candidate and the accomplished candidate is their focus on individual students. The competent candidate attends to “a full range of students” in instruction. This means that intentional instructional strategies are in place that attends to learning needs of different groups of students. For example, attending to the context and or language within a story situation, or encouraging the use of various representations are efforts to ensure that a range of students have access to solving the problem. Grouping structures are implemented in ways that all students are accountable for and supported in their learning. The accomplished candidate, in addition to these strategies, implements instruction in ways that not only has such strategies in place, but also includes ways to hold each student accountable and monitors the progress and learning of each student.

4c) Incorporate Mathematics-Specific Tools:

Candidates select mathematics-specific tools, including technology, to support students’ learning, understanding, and application of mathematics and integrate tools into instruction.

<p>Level 1 The Beginning Candidate</p> <p>Candidate selects tools without regard to supporting students’ learning, understanding, and application of mathematics.</p>	<p>Level 2 The Developing Candidate</p> <p>Candidate selects mathematics-specific tools, including technology, to support students’ learning, understanding, and application of mathematics and is unable or unsuccessful in integrating tools into instruction.</p>	<p>Level 3 The Competent Candidate</p> <p>Candidate selects mathematics-specific tools, including technology, to support a full range of students’ learning, understanding, and application of mathematics and integrates tools into instruction.</p>	<p>Level 4 The Accomplished Candidate</p> <p>Candidate selects mathematics-specific tools, including technology, to support each and every students’ learning, understanding, and application of mathematics and integrates tools into instruction.</p>
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Supporting Explanation:

Candidates must be able to **select** and **incorporate** available mathematics-specific tools that support mathematical reasoning, sense making, and problem solving when doing mathematics themselves and in supporting student learning, understanding, and application of mathematics (NCTM 2014, AMTE, 2017 C.1.6, NGS Center & CCSSO, 2010). Importantly, all students should have access to technology and other tools, such as concrete models, that enhance their learning. Tools should not be presented as a reward or limited to use after mastering selected content or performing paper and pencil algorithms.

Mathematics-specific tools include manipulatives, both physical and virtual. Many tools used in elementary school can support learning in secondary school. For example, geometric growing patterns created with pattern blocks, color tiles, or cubes can support student learning of linear and exponential growth, as well as concepts related to functions. Many manipulatives apply to secondary mathematics, including algebra tiles, geoboards, protractors, compasses, and geometric solids.

Digital technology may serve any or all of these three functions (Drjvers, Boon, & Van Reeuwijk, 2011):

1. tools for doing mathematics (when the learning goal is not to develop computational expertise);
2. fostering the development of conceptual understanding;
3. practicing skills.

Technology can be used for gathering data, enhancing precision, problem solving, predictions, running simulations, and promoting visualization. The platform of the technology is not as important as its functionality. Computers, tablets, smartphones, and advanced calculators can support students in understanding concepts and procedures, and in engaging in mathematical reasoning. (NCTM, 2014). Technology tools that support students' learning, understanding, and applications of mathematics include the following. (NCTM, 2014, p.78-79)

- Graphing applications - can allow students to examine multiple representations of functions and data.
- Spreadsheet applications - can quickly display the results of repeated calculations and generate tables of values using a variety of graphical representations allowing students to develop insights into mathematical structures and relationships
- Computer algebra systems - can operate on algebraic statements
- Interactive geometry applications - allow exploration of geometry conjectures in well-constructed diagrams
- Modeling tools - can be useful in exploring three-dimensional objects
- Data analysis applications - ranging from intuitive tools to tools that support advanced analyses and dynamic representations
- Software applications - including virtual manipulatives

Nonmathematical technologies and tools such as word processing, presentation software, and communication applications can also support interactions in the mathematics classroom. These applications can allow students to share their understanding by communicating their thinking and

receiving constructive feedback and sharing their work outside the classroom. The use of these technologies can also help students get a sense of ownership of the mathematics they are learning. (NCTM, 2014)

It is important for teachers to select and integrate tools and technologies in the classroom to support students' understanding and application of mathematics. The effective use of tools focuses on student conceptual understanding, problem solving and reasoning. Teaching students procedures on how to use tools to solve problems without giving students the opportunity of thinking through the problems and connecting the procedures with formal mathematical reasoning does not allow for students to grow their mathematical thinking.

4d) Use Mathematical Representations:

Candidates select and use mathematical representations to engage students in examining understandings of mathematics concepts and the connections to other representations.

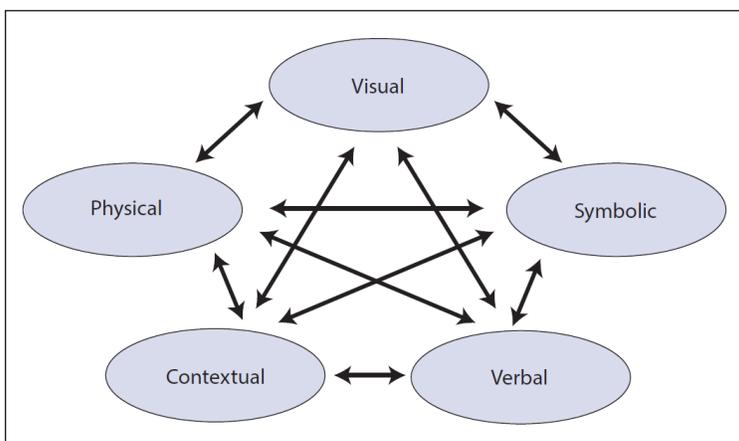
<p>Level 1 The Beginning Candidate</p> <p>Candidate selects mathematical representations without regard to supporting students' learning, understanding and application of mathematics.</p>	<p>Level 2 The Developing Candidate</p> <p>Candidate selects mathematical representations to support students' learning, understanding and application of mathematics and is unable or unsuccessful in implementing or connecting representations during instruction.</p>	<p>Level 3 The Competent Candidate</p> <p>Candidate selects mathematical representations to support students' learning, understanding and application of mathematics and implements and connects representations during instruction.</p>	<p>Level 4 The Accomplished Candidate</p> <p>Candidate selects and connects mathematical representations to support students' learning, understanding and application of mathematics and implements and facilitates students making connections between representations.</p>
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Supporting Explanation:

Candidates understand that all students are capable of thinking mathematically and are able to solve sophisticated mathematical problems with effort (AMTE, 2017, C.1.3). Candidates therefore must provide students with opportunities to approach problems in different ways, using different representations. Candidates must treat representations as essential elements in supporting students' learning, understanding, and application of mathematics.

Mathematical Representations

Mathematical representations embody critical features of mathematical constructs and actions (NCTM, 2014, p. 25). The general classification for mathematical representations includes contextual, visual, verbal, physical, and symbolic representations:



Mathematical representations can be used to understand, learn and apply a mathematical concept in one area of mathematics or to make connections among mathematical content domains. Tripathi (2008) noted that using different representations to solve problems is like “viewing a concept through a variety of lenses, with each lens providing a different perspective that makes the concept richer and deeper.” (p.439). Selecting and using representations to support students’ application of mathematics also include mathematical models. In a mathematical model, representations of a particular real-world problem are used in an attempt to describe, explore, or understand (NCTM, 2018).

Select and Connect Representations

Students should be able to engage in making connections among representations and use them as tools to help them solve problems (NCTM 2014). For this to occur, candidates must be able to first select appropriate representations for the content of a lesson or unit. A competent candidate is able to enact instruction that uses appropriate representations and themselves are able to explain how different representations are connected. This includes explicit statements or questions that require students to connect between representations. The accomplished candidate facilitates instruction so that it is the students that are seeing and making the connections among mathematical representations.

4e) Elicit and Use Student Responses:

Candidates use multiple student responses, potential challenges and misconceptions, and highlight students' thinking as a central aspect of mathematics teaching and learning.

Level 1 The Beginning Candidate	Level 2 The Developing Candidate	Level 3 The Competent Candidate	Level 4 The Accomplished Candidate
<p>Candidate is unable to elicit or use student responses reflecting their thinking to inform instruction.</p>	<p>Candidate elicit multiple student responses reflecting their thinking including potential challenges or misconceptions.</p> <p>Candidates are unable to use student responses to inform the mathematics teaching and learning process.</p>	<p>Candidate elicits multiple student responses, potential challenges and misconceptions.</p> <p>Candidate notices and tracks multiple student responses, as well as challenges or misconceptions as students are solving problems.</p> <p>Candidate uses students' multiple methods and/or challenges and/or misconceptions to engage the full range of students in extending their mathematical learning.</p>	<p>Candidate considers individual and group differences when eliciting multiple student responses, potential challenges and misconceptions.</p> <p>Candidate notices and tracks multiple student responses, as well as challenges or misconceptions as students are solving problems.</p> <p>Candidate uses students' multiple methods and/or challenges and/or misconceptions to engage each and every student in extending their mathematical learning.</p>

Supporting Explanation:

Attending to student thinking is necessary for effective and equitable mathematics instruction, as each individual student has different knowledge bases, representations, interests, and strategies. As described in the *AMTE Standards for the Preparation of Teachers of Mathematics*, “Well-prepared beginners know that the quality and focus of their teaching is affected by the depth and detail of their insight into each student’s mathematical thinking.” (AMTE, 2017, C.3.1) Candidates must be able to draw upon their students’ knowledge and experiences as they engage their students in mathematics lessons. This requires both anticipating student thinking, as well as responding to student thinking.

Implicit in component 4e is the reality that mathematical tasks can almost always be approached in multiple ways, and therefore the teaching of mathematics must focus on multiple approaches. Teaching a topic by telling students to ‘solve it this way’ denies students access to procedural fluency, which includes strategy selection and flexible use of strategies (NRC, 2001). Hence, in order to support students’ emerging procedural fluency and overall mathematical proficiency, candidates must be able to describe different approaches students might use in solving a task as well as present the task in a way that invites students to employ their choice of strategies, representations and tools in solving a problem.

The focus on potential challenges and misconceptions in this component ensures that candidates attend to deep knowledge of the content. For example, in studying functions, a challenge might be that students are not able to relate functions to real life examples, or they are not able to make connections among representations for functions. Further, students may be confused about the relationships between equations written as $y =$ and $f(x) =$. Further, students may or may not have previously learned content that could inform their solving of the task. Anticipating such possibilities should then lead to designing higher-level thinking questions that can help students deepen their understanding.

An important aspect of this component is the focus on the strategic in-the-lesson decision making related to what questions to pose to students to build on their thinking and the thinking of others as they work on tasks, to notice what strategies are being used and what challenges are occurring in order to decide what student work will be shared, how it will be shared, and how that the work be connected back to the key mathematical ideas of the learners.

4f) Develop Conceptual Understanding and Procedural Fluency:

Candidates use conceptual understanding to build procedural fluency for students through instruction that includes explicit connections between concepts and procedures.

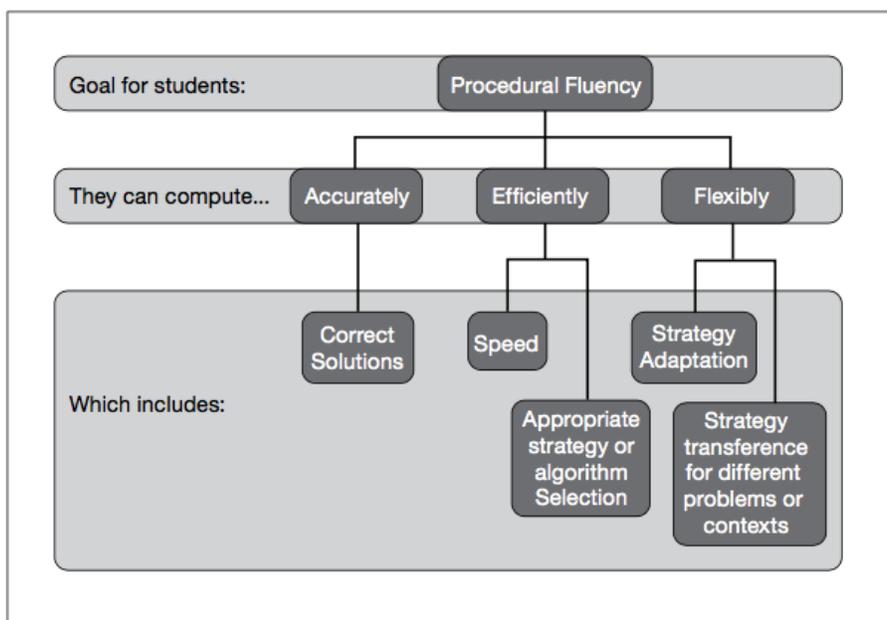
Level 1 The Beginning Candidate	Level 2 The Developing Candidate	Level 3 The Competent Candidate	Level 4 The Accomplished Candidate
Candidate designs instruction that does not include both conceptual understanding and procedural fluency.	Candidate designs instruction that includes both conceptual understanding and procedural fluency, but the conceptual understanding does not serve as a foundation for or is connected to developing procedural fluency.	Candidate designs and implements instruction which uses conceptual understanding to build procedural fluency including explicit connections between concepts and procedures.	Candidate designs and implements instruction which uses conceptual understanding to build procedural fluency including explicit connections between concepts and procedures. Candidate facilitates students making connections between procedures and concepts.

Supporting Explanation:

Candidates must be able to develop mathematical proficiency in all of their students. Kilpatrick, Swafford and Findell (2001) define mathematical proficiency as having five intertwining strands:

- conceptual understanding—an understanding of concepts, operations and relations. This frequently results in students comprehending connections and similarities between interrelated facts.
- procedural fluency—flexibility, accuracy and efficiency in implementing appropriate procedures. Skill in proficiency includes the knowledge of when and how to use procedures. This includes efficiency and accuracy in basic computations.
- strategic competence—the ability to formulate, represent and solve mathematical problems. This is similar to problem solving. Strategic competence is mutually supportive with conceptual understanding and procedural fluency.
- adaptive reasoning—the capacity to think logically about concepts and conceptual relationships. Reasoning is needed to navigate through the various procedures, facts and concepts to arrive at solutions.
- productive disposition—positive perceptions about mathematics. This develops as students gain more mathematical understanding and become capable of learning and doing mathematics.

Importantly, procedural fluency goes well beyond knowing how to implement an algorithm or procedure; such efforts might be referred to as procedural skill. Procedural fluency includes attention to flexibility and strategy selection, as illustrated in the graphic here (Bay-Williams & Stokes, 2017):



This component focuses on teaching such that connections are made between the first two elements, which in turn creates students who have strategic competence, adaptive reasoning, and

productive dispositions. For example, consider how a ‘solve for y’ lesson might be approached, and how a problem like $5(y + 1) = 3(y + 1) + 8$ could be used in such a lesson. In a classroom focused on the skill of determining the value for y, a teacher might ask students to follow a series of steps (eliminate parentheses first, then combine like-terms, etc.). Note that the elements of mathematical proficiency are barely addressed if at all. Imagine instead that students are given the following way to engage with this problem (Star, Kenyon, Joiner, Rittle-Johnson, Evitts, & Heinz, 2010):

Mandy's Solution:	Erica's Solution:
$5(y + 1) = 3(y + 1) + 8$ $5y + 5 = 3y + 3 + 8$ $5y + 5 = 3y + 11$ $2y + 5 = 11$ $2y = 6$ $y = 3$	$5(y + 1) = 3(y + 1) + 8$ $2(y + 1) = 8$ $y + 1 = 4$ $y = 3$
<p style="text-align: right;"><i>Distribute</i></p> <p style="text-align: right;"><i>Combine</i></p> <p style="text-align: right;"><i>Subtract on Both</i></p> <p style="text-align: right;"><i>Subtract on Both</i></p> <p style="text-align: right;"><i>Divide on Both</i></p>	<p style="text-align: right;"><i>Subtract on Both</i></p> <p style="text-align: right;"><i>Divide on Both</i></p> <p style="text-align: right;"><i>Subtract on Both</i></p>

1. Mandy and Erica solved the problem differently, but they got the same answer. Why?
2. Why might you choose to use Erica's way?

Notice that this task requires a deeper conceptual understanding of what a like-term might be, it challenges students to think about different approaches, thereby working on flexibility and strategy selection, and in having the students focus on why both methods work and when they might use Erica's way, they are development strategic competence and adaptive reasoning.

The development of mathematical proficiency takes time, and it requires frequent attention to making connections between procedures and concepts and among procedures. Some lessons may be more heavily focused on conceptual foundations, while other lessons may be focused on choosing among possible procedures to solve a given problem. In any case, asking why procedures work and when they are useful (both higher-level thinking questions) helps students to make sense of the mathematics and supports their emerging proficiency

4g) Facilitate Discourse:

Candidates pose purposeful questions to facilitate discourse among students that ensures that each student learns rigorous mathematics and builds a shared understanding of mathematical ideas.

Level 1 The Beginning Candidate	Level 2 The Developing Candidate	Level 3 The Competent Candidate	Level 4 The Accomplished Candidate
Candidate is unable to pose questions that focus on rigorous learning goals and are not able to facilitate discourse among students in support of building shared understanding of mathematical ideas.	Candidates pose questions that focus students on the rigorous mathematical goals or making connections; or candidate facilitates discourse among students to build shared understanding of mathematical ideas, but discourse is limited to a subset of students.	Candidates pose questions that focus students on the rigorous mathematical goals or making connections. Candidate facilitates discourse among students to build shared understanding of mathematical ideas and ensure that a full range of students engage in rigorous mathematics.	Candidates pose questions that focus students on the rigorous mathematical goals and making connections. Candidate facilitates discourse among students to build shared understanding of mathematical ideas and ensures that each and every student engages in rigorous mathematics.

Supporting Explanation:

Mathematical discourse includes the “purposeful exchange of ideas through classroom discussion, as well as through other forms of verbal, visual, and written communication.” (NCTM, 2014, p. 29). Such discourse allows and expects students to share ideas, clarify understandings, articulate how and why a process or procedures works, when a particular strategy or idea is needed or useful, and make connections among mathematical ideas. Mathematical discussions in high school classrooms too often reflect Level 0 in the as described in the table below, despite recommendations for such discourse as early as the original NCTM standards in 1989. To change this reality, it is imperative that secondary mathematics candidates demonstrate the ability to plan for and to facilitate such discussions.

	Teacher role	Questioning	Explaining mathematical thinking	Mathematical representations	Building student responsibility within the community
Level 0	Teacher is at the front of the room and dominates conversation.	Teacher is only questioner. Questions serve to keep students listening to teacher. Students give short answers and respond to teacher only.	Teacher questions focus on correctness. Students provide short answer-focused responses. Teacher may give answers.	Representations are missing, or teacher shows them to students.	Culture supports students keeping ideas to themselves or just providing answers when asked.
Level 1	Teacher encourages the sharing of math ideas and directs speaker to talk to the class, not to the teacher only.	Teacher questions begin to focus on student thinking and less on answers. Only teacher asks questions.	Teacher probes student thinking somewhat. One or two strategies may be elicited. Teacher may fill in an explanation. Students provide brief descriptions of their thinking in response to teacher probing.	Students learn to create math drawings to depict their mathematical thinking.	Students believe that their ideas are accepted by the classroom community. They begin to listen to one another supportively and to restate in their own words what another student has said.
Level 2	Teacher facilitates conversation between students, and encourages students to ask questions of one another.	Teacher asks probing questions and facilitates some student-to-student talk. Students ask questions of one another with prompting from teacher.	Teacher probes more deeply to learn about student thinking. Teacher elicits multiple strategies. Students respond to teacher probing and volunteer their thinking. Students begin to defend their answers.	Students label their math drawings so that others are able to follow their mathematical thinking.	Students believe that they are math learners and that their ideas and the ideas of their classmates are important. They listen actively so that they can contribute significantly.
Level 3	Students carry the conversation themselves. Teacher only guides from the periphery of the conversation. Teacher waits for students to clarify thinking of others.	Student-to-student talk is student initiated. Students ask questions and listen to responses. Many questions ask “why” and call for justification. Teacher questions may still guide discourse.	Teacher follows student explanations closely. Teacher asks students to contrast strategies. Students defend and justify their answers with little prompting from the teacher.	Students follow and help shape the descriptions of others’ math thinking through math drawings and may suggest edits in others’ math drawings.	Students believe that they are math leaders and can help shape the thinking of others. They help shape others’ math thinking in supportive, collegial ways and accept the same support from others.

Fig. 11. Levels of classroom discourse. From Hufford-Ackles, Fuson, and Sherin (2014), table 1.

The competent candidate and accomplished candidate are able to question and encourage mathematical thinking as described at Level 2. The accomplished candidate is able to orchestrate student discussions in ways that ensure each student is accountable, engaged, and makes contributions to the overall discussion. While every student may not share within a whole class discussion, use of strategies like wait time and think-pair-share ensure that every student’s understanding and contributions matter to the whole class discussion. In both cases, the focus of

the discourse is on the rigorous learning goals, and an outcome of the discourse is that every student has a deeper understanding of those learning goals.

Standard 5: Assessing Impact on Student Learning

Candidates assess and use evidence of students' learning of rigorous mathematics to improve instruction and subsequent student learning. Candidates analyze learning gains from formal and informal assessments for individual students, the class as a whole, and subgroups of students disaggregated by demographic categories, and use this information to inform planning and teaching.

Supporting Explanation:

Well-prepared beginning teachers of mathematics can assess and analyze students' thinking.

“In *Assessment Standards for School Mathematics* (1995), NCTM defined assessments as ‘the process of gathering evidence about a student’s knowledge of, ability to use, and disposition toward, mathematics and of making inferences from that evidence for a variety of purposes’ “(p.3).

“At the same time, NCTM posited that assessment should serve four distinct functions in school mathematics:

- *Monitoring students' progress* to promote student learning.
- *Making instructional decisions* to modify instruction to facilitate student learning
- *Evaluating students' achievement* to summarize and report students' demonstrated understanding at a particular moment in time
- *Evaluating programs* to make decisions about instructional programs

The edTPA glossary indicates that assessment “[R]efer[s] to all those activities undertaken by teachers and by their students . . . that provide information to be used as feedback to modify the teaching and learning activities”¹ for both students and teachers. Assessments provide evidence of students’ prior knowledge, thinking, or learning in order to evaluate what students understand and how they are thinking at a given point in time for the purpose of promoting student learning. Informal assessments may include such things as student questions and responses during instruction and teacher observations of students as they work. Formal assessments may include such things as quizzes, homework assignments, journals, and projects.”

Drawing from the discussion of successful mathematics teaching practices in Standard 4, candidates are provided rich data and evidence of students’ learning. The evaluation of that evidence gives insights into students’ understanding, misconceptions, confidence, hesitation, relationship with the content, and their mathematical identities. The cycle of planning, instruction and assessment is inextricably linked and the candidates’ responsibility for helping

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each student achieve success and develop a positive mathematical identity depends upon a teacher candidate's ability to assess. (NCTM, 2014, p. 89).

AMTE suggests the dynamic nature and the importance of developing skill in teacher candidates to obtain the richest assessment data possible. "They [the teachers] examine their students' varied approaches to mathematical work and respond appropriately. They gather and use information available through daily classroom interactions, routine formative assessments, summaries documenting students' engagement with computer software or tablet applications, summative assessments, and standardized tests. Well-prepared beginners know the affordances and limitations of these sources of data for understanding student thinking and look for patterns across data sources that provide a sound basis for instructional next steps. They have the mathematical knowledge and the inclination to analyze written and oral student productions, looking for each student's mathematical reasoning even when that reasoning may be different from that of the teacher or the student's peers. They also enhance their own observations by deliberately drawing on the insights of families, professional colleagues, and sources of information from beyond the classroom. "(AMTE, 2017)

This standard is intended to cast a broad net as to the type of assessments that a candidate can use. While the standard speaks of formal and informal assessments, it is acknowledged that these will be used both formatively and summatively on a regular basis. While the educational system uses assessment in a variety of ways, the focus of this standard is on the use of assessment to support student learning. Of course, this focus has implications for the teacher candidates as they grow and develop professionally, but the emphasis here is to gather and effectively use as much information as possible to help students succeed and grow in their mathematical expertise.

Assessments include monitoring group problem solving, supporting discussion of understanding, diagnostic quizzes or exercises, pretesting to determine a current level of knowledge, end of quarter or semester assessments, standardized assessments used by schools, districts, and policy makers.

Connecting Components and Indicators on Teaching Meaningful Mathematics

NCTM Component	<i>AMTE Standards for the Preparation of Teachers of Mathematics</i>	<i>NCTM Principles to Actions Mathematics Teaching Practice</i>
<p>5a) Assessing for Learning: Candidates select, modify or create both informal and formal assessments to elicit information on students’ progress toward rigorous mathematics learning goals.</p>	<p>Indicator C.3.1. Anticipate and Attend to Students’ Thinking About Mathematics Content Well-prepared beginning teachers of mathematics anticipate and attend to students’ mathematical thinking and mathematical learning progressions.</p> <p>Indicator C.2.2. Plan for Effective Instruction Well-prepared beginning teachers of mathematics attend to a multitude of factors to design mathematical learning opportunities for students, including content, students' learning needs, students’ strengths, task selection, and the results of formative and summative assessments.</p>	<p>Establish mathematics goals to focus learning. Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions, and uses the goals to guide instructional decisions.</p> <p>Pose purposeful questions. Effective teaching of mathematics uses purposeful questions to assess and advance students’ reasoning and sense making about important mathematical ideas and relationships.</p> <p>Implement tasks that promote reasoning and problem solving. Effective teaching of mathematics engages students in solving and discussing tasks that promote mathematical reasoning and problem solving and allow multiple entry points and varied solution strategies.</p>
<p>5b) Candidates collect information on students’ progress and use data from informal and formal assessments to</p>	<p>Indicator C.1.5. Analyze Mathematical Thinking</p>	<p>Elicit and use evidence of student thinking. Effective teaching of mathematics uses</p>

<p>analyze progress of individual students, the class as a whole, and subgroups of students disaggregated by demographic categories toward rigorous mathematics learning goals.</p>	<p>Well-prepared beginning teachers of mathematics analyze different approaches to mathematical work and respond appropriately.</p>	<p>evidence of student thinking to assess progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning.</p>
<p>5c) Candidates use the evidence of student learning of individual students, the class as a whole, and subgroups of students disaggregated by demographic categories to analyze the effectiveness of their instruction with respect to these groups. Candidates propose adjustments to instruction to improve student learning for each and every student based on the analysis.</p>	<p>Indicator C.2.1. Promote Equitable Teaching</p> <p>Well-prepared beginning teachers of mathematics structure learning opportunities and use teaching practices that provide access, support, and challenge in learning rigorous mathematics to advance the learning of every student.</p>	<p>Elicit and use evidence of student thinking.</p> <p>Effective teaching of mathematics uses evidence of student thinking to assess progress toward mathematical understanding and to adjust instruction continually in ways that support and extend learning.</p>

Selecting Evidence for Standard 5:

Evidence for standard 5 must show the full range of the assessment cycle and demonstrate actual implementation of assessment strategies through analysis of the results of data from work with secondary school students. The evidence can come from field experiences or student teaching/internships.

An assessment of impact on students learning could include student work samples, performance assessments, such as the edTPA, case studies of secondary classrooms, student interviews and classroom action research projects.

5a) **Assessing for Learning:**

Candidates select, modify or create both informal and formal assessments to elicit information on students’ progress toward rigorous mathematics learning goals.

<p>Level 1 The Beginning Candidate Candidate uses informal and/or formal assessments, but assessments do not measure rigorous mathematics learning goals.</p>	<p>Level 2 The Developing Candidate Candidate uses informal or formal assessments to elicit progress towards rigorous mathematics learning goals.</p>	<p>Level 3 The Competent Candidate Candidate selects, creates or adapts assessments and uses both informal and formal assessments to elicit progress towards rigorous mathematics learning goals for a full range of students.</p>	<p>Level 4 The Accomplished Candidate Candidate selects, creates or adapts assessments and uses both informal and formal assessments to elicit progress towards rigorous mathematics learning goals for students’ individual learning needs.</p>
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For teacher candidates, the ability to articulate short and long-term goals for learning is a challenging experience given that their own experience of learning mathematics may not have been one in which such goals were often articulated. “Effective teaching of mathematics establishes clear goals for the mathematics that students are learning, situates goals within learning progressions and sets the goals to guide instructional decisions” (NCTM, 2014, p. 12). Rigorous mathematical goals clearly articulate mathematical learning for both short-term and long-term understanding of mathematics that include ensuring that students have the chance to think deeply, articulate their understanding, use multiple representations, and learn from each other. Goals help enhance learning that reflects conceptual understanding, procedural fluency, and reasoning and problem solving.

Well-articulated goals help determine the type of assessments that provide information as to how they are met. The assessments illuminate not only understandings, but gaps in thinking and misconceptions. For this purpose, the assessments selected are deliberate and purposeful, and intended to provide the most information possible to document student progress toward those goals. They are chosen to help provide insights into individual student performance and can highlight common errors, struggles, and confusions in a way that helps diagnose student progress. In some cases, the assessments can be located in materials and resources. In other cases, the assessments are developed by the candidate or adapted from examples.

5b) Candidates collect information on students’ progress and use data from informal and formal assessments to analyze progress of individual students, the class as a whole, and subgroups of students disaggregated by demographic categories toward rigorous mathematics learning goals.

<p>Level 1 The Beginning Candidate Candidate does not use data from assessments to analyze progress toward rigorous mathematics learning goals.</p>	<p>Level 2 The Developing Candidate Candidate uses data from informal or formal assessments to analyze progress toward rigorous mathematics learning goals for selected students, the class as a whole, or subgroups of students disaggregated by demographic categories.</p>	<p>Level 3 The Competent Candidate Candidate uses data from informal and formal assessments to analyze progress toward rigorous mathematics learning goals for selected students, the class as a whole, and subgroups of students disaggregated by demographic categories, when directed.</p>	<p>Level 4 The Accomplished Candidate Candidate consistently uses data from informal and formal assessments to analyze progress toward rigorous mathematics learning goals for each individual student, the class as a whole, and subgroups of students disaggregated by demographic categories.</p>
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While some data are collected and acted upon immediately in the classroom, the ongoing analysis of other data provides insights into candidate learning based upon the specific learning goals. To assess student progress, candidates study and monitor the progress or difficulties of the class as a whole and individual students. Given that classrooms contain students who come from groups that have been traditionally underserved, underprepared and marginalized in the teaching and learning of mathematics, analysis of progress toward goals for sub-groups of these types of students is valuable and noteworthy. Candidates must show evidence of being prepared to study students' progress in a variety of ways, including attention to the sub-groups of students who may have different learning needs or may progress at different rates and in different ways. Ultimately, candidates assess progress towards goals for the class as a whole, sub-groups and individual students.

Candidates must understand how ELL students, special education students, and reluctant learners can provide evidence of mathematical understanding that may not involve paper and pencil calculations. Systems to track data over time must be developed and utilized. Monitoring progress also assesses the effectiveness of enacted instruction on the students' progress.

5c) Candidates use the evidence of student learning of individual students, the class as a whole, and subgroups of students disaggregated by demographic categories to analyze the effectiveness of their instruction with respect to these groups. Candidates propose adjustments to instruction to improve student learning for each and every student based on the analysis.

<p>Level 1 The Beginning Candidate Candidate does not use evidence of student learning to analyze the effectiveness of their instruction, or they analyzed effectiveness of instruction without proposing adjustments to instruction.</p>	<p>Level 2 The Developing Candidate Candidate uses evidence of student learning to analyze the effectiveness of their instruction and proposes adjustments to instruction, but those adjustments are not explicitly connected to the analysis of the data for selected students, the class as a whole, or subgroups of students disaggregated by demographic categories.</p>	<p>Level 3 The Competent Candidate Candidate uses evidence of student learning to analyze the effectiveness of their instruction and proposes adjustments to instruction that are explicitly connected to the analysis of the data for selected students, the class as a whole, and subgroups of students disaggregated by demographic categories when directed.</p>	<p>Level 4 The Accomplished Candidate Candidate consistently uses evidence of student learning to analyze the effectiveness of their instruction and propose adjustments to instruction that are explicitly connected to the analysis of the data and address the learning needs of each individual student, the class as a whole, and subgroups of students disaggregated by demographic categories without prompting.</p>
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Successful use of student assessment data ensures that the subsequent instruction is tailored to students to promote further understanding. The candidates use of assessment suggests which mathematical teaching practices are effective and ensure that each and every student has access to instruction that supports achievement of rigorous mathematical goals.

As candidates continuously cycle through planning, instruction and assessment, they look for patterns, aberrations, and unexpected performances across data sources to plan subsequent instruction. Looking at data disaggregated by demographics may also provide illuminating evidence regarding similarities and differences in mathematical understanding that suggest how instruction may be differentiated, supplemented or enhanced to assist all students to reach rigorous mathematical goals. This component suggests that assessment data analysis includes looking at trends over time, benchmarking results against past or suggested performances.

Standard 6: Social and Professional Context of Mathematics Teaching and Learning

Candidates are reflective mathematics educators who collaborate with colleagues and other stakeholders to grow “professionally, to support student learning, and to create more equitable mathematics learning environments.

As beginning professionals, candidates must constantly reflect upon their practice, learning from their teaching and the reactions and actions of their students. Critical reflection implies that candidates have planned carefully, learned from their students while teaching a lesson, and begin reflection immediately on how to improve and enhance each and every students’ opportunities to learn. Realizing that they do not work in isolation, the candidates actively seek resources from colleagues and other stakeholders who can provide support and assistance to the candidate informed by their expertise, experience, and knowledge of students.

Beginning candidates who recognize the value of this collaboration also seek out colleagues to ensure that they are providing instruction suited to the needs of each and every student to ensure that students are in a classroom where high quality mathematics learning is accessible to all. Candidates seek counsel to ensure that their practice provides equitable access for all students. Candidates actively seek to collaborate not just mathematics teachers, but ELL, Special Education, counselors, administrators who can provide assistance, coaching, and support.

By high school, it is often challenging for families to provide support for students. The beginning teacher candidate reaches out to families and caretakers, personally and virtually, to keep them informed about the mathematical opportunities afforded the students and to seek input about how to encourage and support students as they learn mathematics. They ensure that families and caretakers realize the necessity of strong mathematical identity for secondary students.

Candidates also reflect upon their teaching and collaborate with colleagues to ensure objective, ongoing evaluation of their beliefs to ensure their teaching remains productive (NCTM, 2014, p.11) and strive to ensure equitable opportunities for their students.

Connecting Components and Indicators on Teaching Meaningful Mathematics

NCTM Component	<i>AMTE Standards for the Preparation of Teachers of Mathematics</i>	<i>NCTM Principles to Actions Mathematics Teaching Practices</i>
<p>6a) Candidates reflect on their impact on students’ mathematical identities and develop professional learning goals that promote students’ positive mathematical identities.</p>	<p>Indicator C.3.3. Anticipate and Attend to Students’ Mathematical Dispositions</p> <p>Well-prepared beginning teachers of mathematics know key facets of students’ mathematical dispositions and are sensitized to the ways in which dispositions may affect students’ engagement in mathematics.</p> <p>Indicator C.4.2. Cultivate Positive Mathematical Identities</p> <p>Well-prepared beginning teachers of mathematics recognize that their roles are to cultivate positive mathematical identities with their students.</p>	<p>Facilitate meaningful mathematical discourse. Effective teaching of mathematics facilitates discourse among students to build shared understanding of mathematical ideas by analyzing and comparing student approaches and arguments.</p> <p>Support productive struggle. Effective teaching of mathematics consistently provides students, individually and collectively, with opportunities and supports to engage in productive struggle as they grapple with mathematical ideas and relationships.</p>
<p>6b) Candidates communicate with families to share and discuss strategies for ensuring the mathematical success of their children.</p>	<p>Indicator C.2.5. Enhance Teaching Through Collaboration with Colleagues, Families, and Community Members</p> <p>Well-prepared beginning teachers of mathematics seek collaboration with other education professionals, parents, caregivers, and community partners to provide the best mathematics learning opportunities for every student.</p>	

<p>6c) Candidates collaborate with colleagues to grow professionally and support student learning of mathematics.</p>	<p>Indicator C.2.5. Enhance Teaching Through Collaboration with Colleagues, Families, and Community Members</p> <p>Well-prepared beginning teachers of mathematics seek collaboration with other education professionals, parents, caregivers, and community partners to provide the best mathematics learning opportunities for every student.</p>	
<p>6d) Candidates seek to create more equitable learning environments by identifying beliefs about teaching and learning mathematics, and associated classroom practices that produce equitable or inequitable mathematical learning for students.</p>	<p>Indicator C.4.1. Provide Access and Advancement</p> <p>Well-prepared beginning teachers of mathematics recognize the difference between access to and advancement in mathematics learning and work to provide access and advancement for every student.</p> <p>Indicator C.4.5. Enact Ethical Practice for Advocacy</p> <p>Well-prepared beginning teachers of mathematics are knowledgeable about, and accountable for, enacting ethical practices that enable them to advocate for themselves and to challenge the status quo on behalf of their students.</p>	

Selecting evidence for Standard 6:

The possibilities for evidence for standard 6 are really open and suggest the flexibility selecting assessments 6-8. The evidence for this standard suggests the documentation of professional development and collaboration, and the documentation and critique of the conditions of schooling. For example, assessments could include analysis and reflection from teaching that highlight professional goals, artifacts that demonstrate collaboration with families and colleagues, or audits of beliefs regarding classroom/school policies that might advocate for better access/achievement for underrepresented groups and students.

6a) Candidates reflect on their impact on students’ mathematical identities and develop professional learning goals that promote students’ positive mathematical identities.

<p>Level 1 The Beginning Candidate Candidate reflects on their impact on students’ mathematical identities but does not develop professional learning goals to better promote students’ positive mathematical identities.</p>	<p>Level 2 The Developing Candidate Candidate reflects on their impact on students’ mathematical identities and develops professional learning goals that promote students’ positive mathematical identities, but without identifying specific strategies or resources.</p>	<p>Level 3 The Competent Candidate Candidate reflects on their impact on students’ mathematical identities and develops professional learning goals that promote students’ positive mathematical identities, including specific strategies for meeting these goals.</p>	<p>Level 4 The Accomplished Candidate Candidate reflects on their impact on individual student’s mathematical identities and develops professional learning goals that promote students’ positive mathematical identities, including specific strategies and professional resources for meeting these goals.</p>
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While candidates reflect on many aspects of their teaching, their role in building positive mathematical identities is explicitly planned (Standard 3). Candidates overtly develop goals related to students' mathematical identities and seek out resources and strategies that assist them in building mathematical confidence and competence of their students. They develop and use opportunities to learn all that they can about their students' perspectives on their own mathematics learning.

“Well-prepared beginners analyze their task selections and implementation, reflecting on the ways they may shape students' mathematical identities. In addition to considering the extent to which the mathematics of the task positions learners as doers, well-prepared beginners consider the contexts of tasks. Contexts such as baseball, rocket launching, or farming may privilege a particular group of students who are familiar and interested in that context. Some classroom practices, such as board races and timed tests, have long-standing history in U.S. classrooms, despite the fact that they exclude those who need more processing time while also communicating that those who are fast are good at mathematics” (AMTE, 2017, Indicator C. 4.2).

6b) Candidates communicate with families to share and discuss strategies for ensuring the mathematical success of their children.

Level 1 The Beginning Candidate	Level 2 The Developing Candidate	Level 3 The Competent Candidate	Level 4 The Accomplished Candidate
<p>Candidate communicates information to families about mathematical ideas and processes.</p>	<p>Candidate communicates information to families about mathematical ideas and processes, and suggests good mathematics resources for families to contribute to the mathematical success of their children.</p>	<p>Candidate communicates with families about the mathematical ideas and processes that students are exploring, suggests good mathematics resources, and provides opportunities for the candidate and families to discuss strategies for ensuring the mathematical success of their children.</p>	<p>Candidate communicates with families about the mathematical ideas and processes that students are exploring, suggests good mathematics resources, and provides opportunities for the candidate and families to discuss strategies for ensuring the mathematical success of their children.</p> <p>Candidate seeks out opportunities in the community to understand and interact with families.</p>

Candidates have professional responsibilities to the communities that they serve. This includes reaching out to families and caretakers to enlist their participation in encouraging and supporting their secondary students as they learn and apply mathematics.

“Too often when a student struggles with mathematics, a parent comments, “I was never very good at math either.” While that may be true, the need for our students to be successful in mathematics is more urgent than at any time in recent history. In this era of focus on college, career, and life readiness, engaging parents is critical to the success of students from prekindergarten through high school. Although parent involvement is an important part of any student’s academic experience, enlisting parent support in mathematics may present a greater challenge and a more conscientious effort on our part. Studies show that many parents are intimidated by coming into schools and meeting teachers—especially mathematics teachers” (NCTM, 2013).

Candidates can encourage families to understand the importance of the mathematics that their students are learning, how to expect and support productive struggle, and how to access assistance from the teacher when the students need it. Technology can assist in communicating with families and caretakers.

6c) Candidates collaborate with colleagues to grow professionally and support student learning of mathematics.

<p>Level 1 The Beginning Candidate</p> <p>Candidate identifies potential collaboration or professional learning opportunities that focus on learning and teaching in mathematics education.</p>	<p>Level 2 The Developing Candidate</p> <p>Candidate collaborates with colleagues or participates in professional development and/or learning communities that focus on learning and teaching in mathematics education.</p>	<p>Level 3 The Competent Candidate</p> <p>Candidate collaborates with colleagues to support student learning of mathematics.</p> <p>Candidate participates in professional development and/or learning communities that focus on learning and teaching in mathematics education.</p>	<p>Level 4 The Accomplished Candidate</p> <p>Candidate identifies opportunities based on targeted professional learning needs;</p> <p>Candidate collaborates with colleagues to support student learning of mathematics;</p> <p>Candidate participates in professional development and/or learning communities that focus on learning and teaching in mathematics education.</p>
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“In education, professionals who are responsible for students’ mathematics learning are never satisfied with their accomplishments and are always working to increase the impact that they have on their students’ mathematics learning... Mathematics teachers are professional who do not do this work in isolation. They cultivate and support a culture of professional collaboration and continual improvement, driven by an abiding sense of interdependence and collective responsibility” (NCTM, 2014, p. 99).

Given the range of curricular and instructional inequities experienced by students, such collaboration can provide support to study options to enhance equity and opportunity for students, to question existing practice, to develop new opportunities for student engagement.

Collaboration to support students learning involves more than the mathematics teachers in the community. There are resources within the school, the district and the broader professional community. Candidates need to be encouraged to avail themselves of such resources including counselors, special education and ELL teachers, coaches, and administrators. Professional organizations, university classes and seminars, and district-organized communities can also provide a systematic structure for such opportunities.

6d) Candidates seek to create more equitable learning environments by identifying beliefs about teaching and learning mathematics, and associated classroom practices that produce equitable or inequitable mathematical learning for students.

Level 1 The Beginning Candidate	Level 2 The Developing Candidate	Level 3 The Competent Candidate	Level 4 The Accomplished Candidate
<p>Candidate is unable to identify beliefs and practices that produce inequitable mathematical learning experiences and outcomes for students.</p>	<p>Candidate identifies beliefs and classroom practices that produce inequitable mathematical learning experiences and outcomes for students.</p> <p>Candidate identifies beliefs that produce equitable mathematical learning experiences and outcomes for students.</p>	<p>Candidate identifies beliefs and classroom practices that produce equitable and inequitable mathematical learning experiences and outcomes for students.</p> <p>Candidate seeks out information to increase equitable practices and/or eliminate inequitable practices to further mathematical learning.</p>	<p>Candidate identifies personal beliefs, classroom practices, and systemic structures that produce equitable and inequitable mathematical learning experiences and outcomes for students.</p> <p>Candidate seeks out information to increase equitable practices and/or eliminate inequitable practices to further mathematical learning for individual students.</p> <p>Candidate describes ways to help traditionally marginalized students experience success.</p>

“An excellent mathematics program requires that all students have access to a high-quality mathematics curriculum, effective teaching and learning, high expectations, and the support and resources needed to maximize their learning potential” (NCTM, 2014, p. 59).

Teacher candidates need overt opportunities to examine both productive and unproductive beliefs about teaching mathematics (NCTM, 2014, p.11) and productive and unproductive beliefs about access and equity in mathematics (NCTM, 2014, p. 63). This exploration often challenges many of the teaching experiences candidates have themselves or understand were experiences of students who struggled or did not have access to equitable access in their own journey to learn mathematics. See Equitable Teaching Practices Crosswalk, Table X, in Standard 4.

Candidates must be able to articulate how equitable and effective mathematics teaching practices can be implemented to support every student and have opportunities to implement those practices during the student teaching experience.

NCTM’s Professionalism Principle states “In an excellent mathematics program, educators hold themselves and their colleagues accountable for the mathematical success of every student and for personal and collective professional growth toward effective teaching and learning of mathematics (NCTM, 2014, p. 99).

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